1. To convert between Fahrenheit, F, and Celsius, C, the formula is $F = \frac{9}{5}C + 32$. Jennifer, having no time to be this precise, instead approximates the temperature of Fahrenheit, $\widehat{F}$, as $\widehat{F} = 2C + 30$. There is a range of temperatures $C_1 \le C \le C_2$ such that for any C in this range, $|\widehat{F} - F| \le 5$. Compute the ordered pair (C_1, C_2).

   Answer: $(-15, 35)$

   Solution: We require $|\hat{F} - F| = |2C + 30 - \frac{9}{5}C - 32| = |\frac{C}{5} - 2| \le 5$. Expanding out the absolute values, this is $-5 \le \frac{C}{5} - 2 \le 5$ or $-15 \le C \le 35$, giving the answer of $\boxed{(-15, 35)}$.

2. Compute integer x such that $x^{23} = 27368747340080916343$.

   Answer: 7

   Solution: 27368747340080916343 has less than 23 digits, so we know that its 23rd root is less than 10. The number is odd, so the 23rd root must be odd. 1^{23} ends in 1, 3^{23} ends in 7, 5^{23} ends in 5, 7^{23} ends in 3, and 9^{23} ends in 9. Thus, the 23rd root of 27368747340080916343 must be $\boxed{7}$.

3. The number of ways to flip n fair coins such that there are no three heads in a row can be expressed with the recurrence relation

$$S(n+1) = a_0 S(n) + a_1 S(n-1) + ... + a_k S(n-k)$$

   for sufficiently large n and k where $S(n)$ is the number of valid sequences of length n. What is

$$\sum_{n=0}^{k} |a_n|?$$

   Answer: 3

   Solution 1: We can consider the different endings of a sequence of length $n+1$. If it ends in T, then there are n remaining spots we can choose in $S(n)$ ways. If it ends in H with the last two tosses being TH, then there are $n-1$ remaining spots to be chosen in $S(n-1)$ ways. Finally, if it ends in H with the last two tosses being HH, we must have THH since HHH results in three heads in a row, so there are $n-2$ remaining spots to be chosen in $S(n-2)$ ways. Thus, we have

$$S(n+1) = S(n) + S(n-1) + S(n-2)$$

   so $a_0 = 1, a_1 = 1, a_2 = 1, a_3 = a_4 = ... = a_k = 0$ and

$$\sum_{n=0}^{k} |a_n| = 1 + 1 + 1 + 0 + 0 + ... + 0 = \boxed{3}.$$

   Solution 2: We can start with a valid sequence of length n and consider how to build up to a sequence of length $n+1$. We know that adding a tail will never result in the sequence becoming invalid, as we started with a valid sequence. However, adding H might result in the sequence being invalid, which happens if and only if the prior sequence ended with HH. This also means that the third to last entry must be a T, as otherwise, the sequence would have a HHH. Then, the $n-3$ remaining slots can be filled in $S(n-3)$ ways, so there are $S(n) - S(n-3)$ ways to count in this case. Overall, we have

$$S(n+1) = 2S(n) - S(n-3)$$

so $a_0 = 2, a_1 = 0, a_2 = 0, a_3 = -1, a_4 = ... = a_k = 0$ and

$$\sum_{n=0}^{k} |a_n| = 2 + 0 + 0 + 1 + 0 + ... + 0 = \boxed{3}.$$

4. For how many three-digit multiples of 11 in the form $\underline{abc}$ does the quadratic $ax^2 + bx + c$ have real roots?

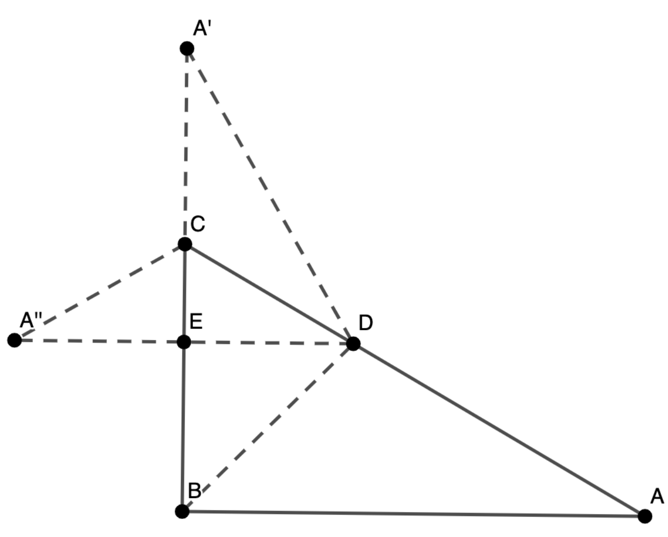
   Answer: 45

   Solution: We consider $11 \times 10, 11 \times 11, \ldots, 11 \times 90$. Note that if $11n$ does not require any carrying in computing $10n + n$, then the tens digits will be the sum of the hundreds and ones digits, and so the quadratic will have real roots. Otherwise, the tens digit will be less than both the hundreds and ones digits, so the roots have nonzero imaginary parts. Thus, our answer is the number of numbers among $10, 11, \ldots 90$ whose digits sum to at most 9. This gives us $\boxed{45}$.

5. William draws a triangle $\triangle ABC$ with $AB = \sqrt{3}, BC = 1$, and $AC = 2$ on a piece of paper and cuts out $\triangle ABC$. Let the angle bisector of $\angle ABC$ meet AC at point D. He folds $\triangle ABD$ over BD. Denote the new location of point A as A'. After William folds $\triangle A'CD$ over CD, what area of the resulting figure is covered by three layers of paper?

   Answer: $\mathbf{\frac{2\sqrt{3}-3}{4}}$

   Solution: The part of the figure that is covered by three layers of paper is $\triangle CDE$ as shown in the diagram. This is a 30-60-90 triangle with hypotenuse CD. Using the angle bisector theorem, we have $CD = \frac{2}{1+\sqrt{3}}$. We compute $CE = CD \cdot \sin(30) = \frac{2}{1+\sqrt{3}} \cdot \frac{1}{2} = \frac{1}{1+\sqrt{3}}$. Similarly, $DE = CD \cdot \cos(30) = \frac{2}{1+\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{1+\sqrt{3}}$. Then, the area can be computed as $\frac{1}{2} \cdot CE \cdot DE = \frac{1}{2} \cdot \frac{1}{1+\sqrt{3}} \cdot \frac{\sqrt{3}}{1+\sqrt{3}} = \boxed{\frac{2\sqrt{3}-3}{4}}$.



6. Compute $(1)(2)(3) + (2)(3)(4) + \ldots + (18)(19)(20)$.

   Answer: 35910

Solution: In summation notation, this expression is

$$\sum_{i=2}^{19}(i-1)(i)(i+1) = \sum_{i=2}^{19} i^3 - i.$$

Since $1^3 - 1 = 0$, this expression is equivalent to

$$\sum_{i=1}^{19} i^3 - i = \sum_{i=1}^{19} i^3 - \sum_{i=1}^{19} i.$$

Since the sum of the first n cubes is $\left(\frac{n(n+1)}{2}\right)^2$ and the sum of the first n positive integers is $\frac{n(n+1)}{2}$, the sum is equal to

$$\left(\frac{19(20)}{2}\right)^2 - \frac{19(20)}{2} = 190^2 - 190 = 190 \cdot 189 = \boxed{35910}.$$

7. An ant starts at the point $(0,0)$. It travels along the integer lattice, at each lattice point choosing the positive x or y direction with equal probability. If the ant reaches $(20,23)$, what is the probability it did not pass through $(20,20)$?

   Answer: $\mathbf{\frac{1510}{1763}}$

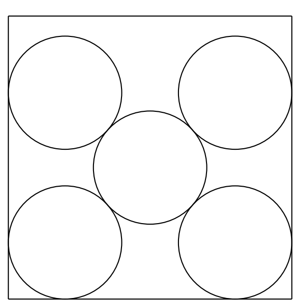
   Solution: There are $\binom{20+23}{20}$ ways for the ant to reach $(20,23)$, and among these, $\binom{20+20}{20}\binom{3}{0}$ ways it could have done so by passing through $(20,20)$. Then, the probability the ant does not pass through $(20,20)$ is $1 - \frac{\binom{40}{20}}{\binom{43}{20}} = \boxed{\frac{1510}{1763}}$.

8. Let $a_0 = 2023$ and a_n be the sum of all divisors of a_{n-1} for all $n \geq 1$. Compute the sum of the prime numbers that divide a_3.

   Answer: 12

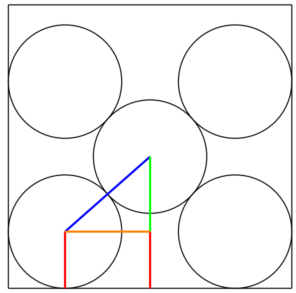
   Solution: We just need to use the formula for finding the sum of all divisors and geometric series. Namely, $2023 = 7 \cdot 17^2$ so $a_1 = (1+7)(1+17+17^2) = 8 \cdot 307$. Then, $a_2 = (1+2+2^2+2^3)(1+307) = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. Repeating this, $a_3 = (1+2+2^2)(1+3)(1+5)(1+7)(1+11) = 2^8 \cdot 3^2 \cdot 7$. So, the answer is $2+3+7 = \boxed{12}$.

9. Five circles of radius one are stored in a box of base length five as in the following diagram. How far above the base of the box are the upper circles touching the sides of the box?



   Answer: $\mathbf{1+\sqrt{7}}$

   Solution:



The red lines, the length of the radius of the circles, are of length 1. The orange line goes from the center of the square to the midline of the square, and is of length $\frac{5}{2} - 1 = \frac{3}{2}$. The blue line connects the centers of two tangential circles and is of length $1 + 1 = 2$. By Pythagorean theorem, the green line is of length $\sqrt{(2^2 - (\frac{3}{2})^2)} = \sqrt{\frac{7}{4}}$. By symmetry, the point of tangency is then $1 + \sqrt{\frac{7}{4}} + \sqrt{\frac{7}{4}} = \boxed{1 + \sqrt{7}}$.

10. Three rectangles of dimension $X \times 2$ and four rectangles of dimension $Y \times 1$ are the pieces that form a rectangle of area $3XY$ where X and Y are positive, integer values. What is the sum of all possible values of X?

Answer: 6

Solution: The area of the rectangles with dimension $X \times 2$ is $2X$, and the area of the rectangles with dimension $Y \times 1$ is Y. Hence we can create the following equation from the information about the areas,

$$3(2X) + 4(Y) = 3XY.$$

This can be equivalently expressed as $3XY - 6X - 4Y = 0$. Using Simon's Favorite Factoring Trick, we factor the previously stated equation as

$$\begin{aligned} X(3Y-6) - 4Y &= 0 \\ X(3Y-6) - \frac{4}{3}(3Y-6) - \frac{4}{3} \cdot 6 &= 0 \\ (X - \frac{4}{3})(3Y-6) &= 8 \\ (3X-4)(3Y-6) &= 24. \end{aligned}$$

To find the values of X and Y, we factor 24 and use these values for the product of $3X - 4$ and $3Y - 6$. Let $a = 3X - 4$ and $b = 3Y - 6$ such that $ab = 24$. For X and Y to be integers, we determine that $a \equiv 2 \pmod 3$ and $b \equiv 0 \pmod 3$. For all the pairs of (a, b) which are factors of 24, we can determine that the only ones which satisfy the previous congruences are $(8, 3)$ and $(2, 12)$ and the corresponding pairs of (X, Y) are $(4, 3)$ and $(2, 6)$, respectively. This results in our answer of $4 + 2 = \boxed{6}$.

11. Suppose we have a polynomial $p(x) = x^2 + ax + b$ with real coefficients $a + b = 1000$ and $b > 0$. Find the smallest possible value of b such that $p(x)$ has two integer roots.

Answer: 852

Solution: Let x, y be the roots of our polynomial. By Vieta's, we have $x + y = -a$ and $xy = b$. Then, $xy - (x + y) = 1000$ so $(x - 1)(y - 1) = 1001 = 7 \cdot 11 \cdot 13$. We want to minimize $xy = 1000 + (x + y)$, so we want to minimize $(x - 1) + (y - 1) = x + y - 2$. When the product is fixed, the sum of two positive integers is the smallest when their difference is the smallest. However, we can make $x-1, y-1$ both negative so we actually want to maximize the absolute value. The best choice would have been $x-1=-1$ and $y-1=-1001$, but this makes $b=0$, so we can only have $x-1=-7$ and $y-1=-143$. Then, we get $xy=(-6)(-142)=\boxed{852}$.

12. Ten square slips of paper of the same size, numbered $0, 1, 2, ..., 9$, are placed into a bag. Four of these squares are then randomly chosen and placed into a two-by-two grid of squares. What is the probability that the numbers in every pair of blocks sharing a side have an absolute difference no greater than two?

Answer: $\mathbf{\frac{1}{90}}$

Solution: Assume we have a working square. Then note that by the triangle inequality, the maximum distance between two numbers is 4, achieved between two numbers on a diagonal. In that case, the other two numbers are equal, which is impossible. It follows that the maximum distance is 3. However, since the square has four distinct numbers, the maximum distance is at least 3. It follows that the maximum distance is 3. That is, the four numbers are consecutive. Given a set of four consecutive numbers $a, a+1, a+2, a+3 \in \{0, 1, ..., 9\}$, note that a and $a+3$ must be diagonally opposite each other on the grid of squares. There are 4 choices for where to place a and 2 choices for where to place $a+1$. The positions for $a+2$ and $a+3$ are then already determined. There are 7 possibilities for the value of a. The answer is thus

$$\frac{8 \cdot 7}{10 \cdot 9 \cdot 8 \cdot 7} = \boxed{\frac{1}{90}}.$$

13. Let $\triangle ABC$ be an equilateral triangle with side length 1. Let the unit circles centered at A, B, and C be Ω_A, Ω_B, and Ω_C, respectively. Then, let Ω_A and Ω_C intersect again at point D, and Ω_B and Ω_C intersect again at point E. Line BD intersects Ω_B at point F where F lies between B and D, and line AE intersects Ω_A at G where G lies between A and E. BD and AE intersect at H. Finally, let CH and FG intersect at I. Compute IH.

Answer: $\mathbf{\frac{3-\sqrt{3}}{6}}$

Solution: Consider the symmetries in the perpendicular bisector of AB. C, I, H lie on this perpendicular bisector. Moreover, Ω_A and Ω_B must also intersect on this perpendicular bisector, let us call this point K. Then, CK, AE, and BD all intersect at H. It follows from symmetry that H must therefore be the centroid of ABC. Therefore, $\angle HAB = \angle GAB = \angle EAB = 30°$. Then let J be the midpoint of AB. It follows that the distance IH is equal to IJ - HJ. We have from the above that the length $HJ = \frac{1}{2\sqrt{3}}$. Then $AH = \frac{1}{\sqrt{3}}$. Since $AG = AH + HG = 1$, it follows that $HG = 1 - \frac{1}{\sqrt{3}}$. Using the similarity of AHJ and HIG, it follows that $IH = \frac{1}{2\sqrt{3}}\left(\frac{1-\frac{1}{\sqrt{3}}}{\frac{1}{\sqrt{3}}}\right) = \frac{1}{2}\left(1-\frac{1}{\sqrt{3}}\right) = \boxed{\frac{3-\sqrt{3}}{6}}$.

14. Suppose Bob randomly fills in a 45×45 grid with the numbers from 1 to 2025, using each number exactly once. For each of the 45 rows, he writes down the largest number in the row. Of these 45 numbers, he writes down the second largest number. The probability that this final number is equal to 2023 can be expressed as $\frac{p}{q}$ where p and q are relatively prime positive integers. Compute the value of p.

Answer: 990

Solution: In order for 2023 to be the final number, we must have 2024 and 2025 in the same row and 2023 in a different row. We can first place 2025 anywhere in the grid. We then have

44 places to put 2024 in the same row as 2025. For 2023, we must put it in a different row, so there are $45 \cdot 44$ possibilities. For the probability we get $\frac{44}{2024} \cdot \frac{45 \cdot 44}{2023} = \frac{2^2 \cdot 11 \cdot 3^2 \cdot 5 \cdot 2^2 \cdot 11}{2^3 \cdot 11 \cdot 23 \cdot 7 \cdot 17^2} = \frac{990}{23 \cdot 7 \cdot 17^2}$, so our answer is $\boxed{990}$.

15. f is a bijective function from the set $\{0, 1, 2, \cdots, 11\}$ to $\{0, 1, 2, \cdots, 11\}$, with the property that whenever a divides b, $f(a)$ divides $f(b)$. How many such f are there? *A bijective function maps each element in its domain to a distinct element in its range.*

Answer: 2

Solution: For any a, $a \mid 0$, so any $f(a) \mid f(0)$, which gives us $f(0) = 0$. For any b, $1 \mid b$, $f(1) \mid f(b)$ for any b, so $f(1) = 1$. Similarly, $f(2) = 2$, $f(3) = 3$ since we get that $f(2)$ must divide 6 elements in the range and $f(3)$ must divide 4 elements in the range. This then forces $f(6) = 6$ since 2 and 3 must divide $f(6)$. $f(5)$ must divide $f(10)$, giving the possibilities $(4, 8)$, $(5, 10)$. In the former case, this forces $f(4) = 5$, $f(8) = 10$, contradicting the given condition in the case $(a, b) = (2, 4)$. In the latter case, $f(4) = 4, f(8) = 8$, and $f(9) = 9$ since it is divisible by 3. The only freedom we are afforded is that we can exchange 7 and 11 if we want to, giving $\boxed{2}$ distinct functions.

16. When not writing power rounds, Eric likes to climb trees. The strength in his arms as a function of time is $s(t) = t^3 - 3t^2$. His climbing velocity as a function of the strength in his arms is $v(s) = s^5 + 9s^4 + 19s^3 - 9s^2 - 20s$. At how many (possibly negative) points in time is Eric stationary?

Answer: 9

Solution: First, factor $v(s) = (s+5)(s+4)(s+1)(s)(s-1)$. This shouldn't be too bad as s is clear, and $s+1, s-1$ can be found by plugging in $1, -1$ which is a common first thing to try. Then, $v(s) = 0$ for $s = -5, -4, -1, 0, 1$.

Now, consider the function $s(t) = t^3 - 3t^2$. This function increases up to $(0, 0)$, then decreases to $(2, -4)$, and then increases. For $s = -5$, this is below the critical points so only one value of t makes $s(t) = -5$. For $s = -4$, there is a value of t in $(-\infty, 0)$ that makes $s(t) = -4$ and $t = 2$ works. For $s = -1$, there are three values, one in $(-\infty, 0)$; one in $(0, 2)$; and one in $(2, \infty)$. For $s = 0$ there is $t = 0$ and some value in $(2, \infty)$. Finally, only one value of t in $(2, \infty)$ makes $s(t) = 1$.

As $s(t)$ is a function, no values of t are double counted. As such, there are $1 + 2 + 3 + 2 + 1 = \boxed{9}$ values of t that make Eric stationary.

17. Consider a triangle $\triangle ABC$ with angles $\angle ACB = 60°$, $\angle ABC = 45°$. The circumcircle around $\triangle ABH$, where H is the orthocenter of $\triangle ABC$, intersects BC for a second time in point P, and the center of that circumcircle is O_c. The line PH intersects AC in point Q, and N is center of the circumcircle around $\triangle AQP$. Find $\angle NO_cP$.

Answer: 45°

Solution: We have that both N and O_C lie on the perpendicular bisector of AP, which intersects AP at a point we denote with M. We will first prove that NO_cBP is cyclic. To do so, note that

$$\angle CQP = 180° - \angle AQP = \frac{1}{2}\angle ANP = \angle MNP,$$

since N is the circumcenter. We then have that $\angle CQP = \angle CAH + \angle QHA$ and $\angle CAH = 90° - \gamma$ and $\angle QHA = 180° - \angle AHP = \angle ABP = \beta$ where $\beta = \angle ABC$ and $\gamma = \angle ACB$. We hence have

$\angle MNP = \angle CQP = 90^\circ - \gamma + \beta$. However, $\angle AHB = 180^\circ - \gamma$ and therefore $\angle AO_cB = 2\gamma$, and so, $\angle O_cBP = 90^\circ - \gamma$. This gives us $\angle O_cBP = 90^\circ - \gamma + \beta = \angle MNP$, and therefore the quadrilateral NO_cBP is cyclic.

We now compute $\angle PO_cB$. Since $O_cP = O_cB$, and $\angle O_cBP = 90^\circ - \gamma + \beta$, we get that $\angle PO_cB = 180^\circ - 2\angle O_cBP = 2(\gamma - \beta)$. On the other hand, we can calculate $\angle NPC$ as follows: $\angle HAP = \angle HBP = 90^\circ - \gamma$, and therefore $\angle APC = \gamma$. Meanwhile, $\angle NPA = 90^\circ - \angle MNP = \gamma - \beta$. As such, $\angle NPC = \gamma + \gamma - \beta = 2\gamma - \beta$. Therefore, $\angle NO_cB = 2\gamma - \beta$, and since $\angle PO_cB = 2\gamma - 2\beta$, we get that $\angle NO_cP = \beta = \boxed{45^\circ}$, as desired. Note that this also means that $N \in AB$.

18. If x, y are positive real numbers and $xy^3 = \frac{16}{9}$, what is the minimum possible value of $3x + y$?

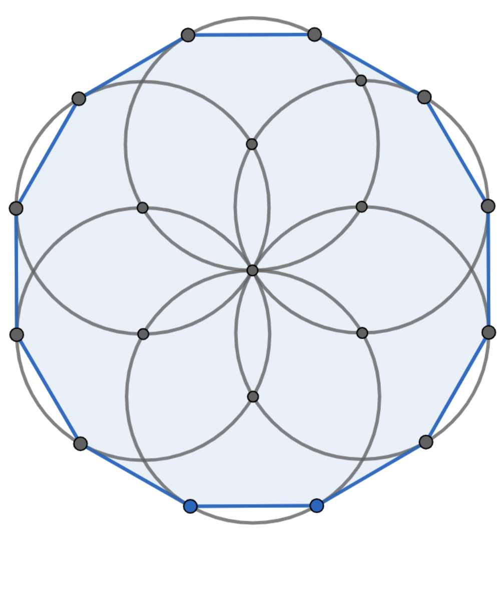
Answer: $\mathbf{\frac{8}{3}}$

Solution: By the AM-GM inequality, we have $\frac{3x + \frac{y}{3} + \frac{y}{3} + \frac{y}{3}}{4} \geq \sqrt[4]{\frac{xy^3}{9}} = \sqrt[4]{\frac{16}{81}} = \frac{2}{3}$. Then, $3x + y \geq 4 \cdot \frac{2}{3} = \boxed{\frac{8}{3}}$.

19. $A_1A_2...A_{12}$ is a regular dodecagon with side length 1 and center at point O. What is the area of the region covered by circles $(A_1A_2O), (A_3A_4O), (A_5A_6O), (A_7A_8O), (A_9A_{10}O)$, and $(A_{11}A_{12}O)$? (ABC) *denotes the circle passing through points* A, B, *and* C.

Answer: $\mathbf{2\pi + 3\sqrt{3}}$

Solution: Consider circle (A_1A_2O). We have $\angle A_1OA_2 = 30^\circ$ and $A_1A_2 = 1$. Let the center of (A_1A_2O) be O'. Then $\angle A_1O'A_2 = 60^\circ$ and $\triangle A_1O'A_2$ is isosceles. Thus, the radius of (A_1A_2O) is 1. The total area of the 6 circles is then 6π. There are 6 regions where two circles intersect and 6 regions where three circles intersect.



The regions where two circles intersect are each formed by two circles that are at an angle of 60° from each other about the center of the dodecagon. Then, a region consists of two equilateral triangles of side length 1 and four sectors of area $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$, so the area of each region is $2 \cdot \frac{\sqrt{3}}{4} + 4\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$.

The regions where two circles intersect are each formed by two circles that are at an angle of $120°$ from each other about the center of the dodecagon. Then, a region consists of two sectors whose total area is $2\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right)=\frac{\pi}{3}-\frac{\sqrt{3}}{2}$.

The regions where two circles intersect are counted twice and the regions where three circles intersect are counted three times if we simply add the areas of the 6 circles. Subtracting the regions where two circles intersect also subtracts each region where three circles intersect twice so that each region is included once, as desired.

The area of the region then is $6\pi-6\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)=\boxed{2\pi+3\sqrt{3}}$.

20. Let $N=2000\ldots0x0\ldots00023$ be a 2023-digit number where the x is the 23rd digit from the right. If N is divisible by 13, compute x.

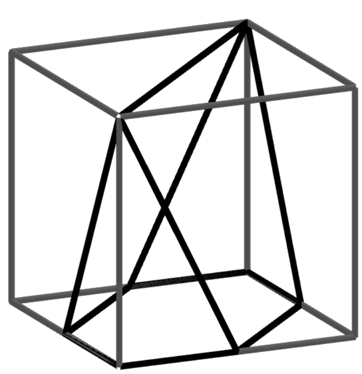
Answer: 9

Solution: Note that $1001=7\times11\times13$, so $1000\equiv-1 \pmod{13}$. We can thus compute $N \pmod{13}$ by considering spans of 3 digits. In order words, if $N=a_1\cdot10^0+a_2\cdot10^3+a_3\cdot10^6+a_4\cdot10^9+\ldots$, then $N\equiv a_1-a_2+a_3-a_4+\ldots \pmod{13}$. For the given N, we have $N\equiv 23-10x+2 \pmod{13}$. This gives us $10x\equiv25 \pmod{13}$, which we can solve to get $x\equiv9 \pmod{13}$. Our answer then is $\boxed{9}$.

21. Alice and Bob each visit the dining hall to get a grilled cheese at a uniformly random time between 12PM and 1PM (their arrival times are independent) and, after arrival, will wait there for a uniformly random amount of time between 0 and 30 minutes. What is the probability that they will meet?

Answer: $\mathbf{\frac{5}{12}}$

Solution: Note that the amount of time the second person waits does not matter—we only care whether the first person to arrive waits long enough for the second person to arrive. If Alice and Bob get to the dining hall x minutes apart, the probability they meet is $\frac{\max(0,30-x)}{30}$. The probability they meet can be represented as a polyhedron contained within the unit cube, where the base of the cube is the probability space of their arrival times and the height represents the probability of meeting for each arrival time. The volume of this polyhedron can be computed by removing the truncated triangular pyramids from each side of the cube. The volume of each truncated triangular pyramid is $\frac{7}{8}\cdot\frac{1}{3}\cdot\frac{1}{2}=\frac{7}{24}$, so our answer is $1-2\cdot\frac{7}{24}=\boxed{\frac{5}{12}}$.



Alternatively, we can solve for the probability by taking an integral along the diagonal of the base of the cube. If the base is the unit square, we integrate from $(\frac{3}{4}, \frac{1}{4})$ to $(\frac{1}{2}, \frac{1}{2})$ to get half of the probability, letting the variable of integration represent the distance from $(1, 0)$. We multiply the probability at each diagonal by the length of the diagonal to get

$$\int_{\frac{\sqrt{2}}{4}}^{\frac{\sqrt{2}}{2}} (2\sqrt{2}x - 1)(2x)\ dx = \frac{5}{24}.$$

Again, we get $\boxed{\frac{5}{12}}$.

22. Consider the series $\{A_n\}_{n=0}^{\infty}$, where $A_0 = 1$ and for every $n > 0$,

$$A_n = A_{\left[\frac{n}{2023}\right]} + A_{\left[\frac{n}{2023^2}\right]} + A_{\left[\frac{n}{2023^3}\right]},$$

where $[x]$ denotes the largest integer value smaller than or equal to x. Find the $(2023^{3^2} + 20)$-th element of the series.

Answer: 653

Solution: We partition the natural numbers into sets $S_k = \{2023^k, \ldots, 2023^{k+1} - 1\}$. Notice that for every $n > 2023, n \in S_k$, $[\frac{n}{2023}] \in S_{k-1}$, and similarly for the other fractions, $\left[\frac{n}{2023^2}\right] \in S_{k-2}$ and $\left[\frac{n}{2023^3}\right] \in S_{k-3}$ (provided these sets exist, i.e. $k > 3$). It is therefore clear that for every $n \in S_k$, the value of A_n is constant. It is therefore sufficient to only look at values of n of the type $n = 2023^k$. We denote $B_k := A_{2023^k}$

Considering $n = 1$, we get $A_1 = 3A_0 = 3$, and therefore, $B_0 = 3$. We can write the recurrence equation for B as

$$B_k = B_{k-1} + B_{k-2} + B_{k-3}.$$

Therefore, we can calculate the values up to B_9 as follows: $B_1 = 3+1+1 = 5, B_2 = 5+3+1 = 9$, etc. Notice that this is a generalization of the Fibonacci sequence. Upon further calculations, we get $B_9 = 653$, and so $A_{2023^9} = 653$, and since A is constant on the set S_9, we get the final answer for $n = 2023^{3^2} + 20, A_n = \boxed{653}$.

23. The side lengths of triangle $\triangle ABC$ are 5, 7 and 8. Construct equilateral triangles $\triangle A_1BC$, $\triangle B_1CA$, and $\triangle C_1AB$ such that A_1, B_1, C_1 lie outside of $\triangle ABC$. Let A_2, B_2, and C_2 be the centers of $\triangle A_1BC$, $\triangle B_1CA$, and $\triangle C_1AB$, respectively. What is the area of $\triangle A_2B_2C_2$?

Answer: $\frac{43\sqrt{3}}{4}$

Solution: Note that $\triangle AB_2C_2$ is similar to $\triangle AB_1B$ by SAS with a ratio of $1 : \sqrt{3}$ and $\triangle CA_2B_2$ is similar to $\triangle CBB_1$ by SAS with a ratio of $1 : \sqrt{3}$. Then, $B_2C_2 = A_2B_2$. Using a similar argument for another pair of sides, we can see that $\triangle A_2B_2C_2$ is equilateral.
Now, we have $5^2 + 8^2 - 5 \cdot 8 = 7^2$, so the angle between the sides of lengths 5 and 8 is $60°$ by the law of cosines. Let this angle be at vertex A. Then, $\angle AB_2C_2 = 120°$. Using the law of cosines, we have $B_2C_2^2 = \left(\frac{5}{\sqrt{3}}\right)^2 + \left(\frac{8}{\sqrt{3}}\right)^2 + \frac{5}{\sqrt{3}} \cdot \frac{8}{\sqrt{3}} = \frac{129}{3} = 43$. Then, the area of $\triangle A_2B_2C_2$ is $\boxed{\frac{43\sqrt{3}}{4}}$.

24. There are 20 people participating in a random tag game around an 20-gon. Whenever two people end up at the same vertex, if one of them is a tagger then the other also becomes a tagger. A round consists of everyone moving to a random vertex on the 20-gon (no matter where they were

at the beginning). If there are currently 10 taggers, let E be the expected number of untagged people at the end of the next round. If E can be written as $\frac{a}{b}$ for a, b relatively prime positive integers, compute $a+b$.

Answer: $\mathbf{19^{10} + 2 \cdot 20^9}$

Solution: Let X be the random variable for the number of different vertices that the 10 taggers end up occupying. Then, each of the untagged people has a $\frac{X}{20}$ chance of ending up at a vertex with a tagged person. So, by linearity of expectation, the expected number of additional taggers is $10 \cdot \frac{X}{20} = \frac{X}{2}$.

Therefore, it remains to compute $\mathbb{E}[X]$. For each of the 20 vertices, have an indicator whether there is at least one tagger on that vertex. By complementary counting, the probability there is at least one is $1 - \left(1 - \frac{1}{20}\right)^{10}$. So, $\mathbb{E}[X] = 20 - 20\left(1 - \frac{1}{20}\right)^{10}$ and the total number of taggers is then $10 + \frac{1}{2}\left[20 - 20\left(1 - \frac{1}{20}\right)^{10}\right] = 20 - 10\left(1 - \frac{1}{20}\right)^{10}$. Therefore, the expected number of untagged is now $10\left(1 - \frac{1}{20}\right)^{10}$. Simplifying this out yields that our answer is $\boxed{19^{10} + 2 \cdot 20^9}$.

25. You are given that 1000! has 2568 decimal digits. Call a permutation π of length 1000 good if $\pi(2i) > \pi(2i-1)$ for all $1 \le i \le 500$ and $\pi(2i) > \pi(2i+1)$ for all $1 \le i \le 499$. Let N be the number of good permutations. Estimate D, the number of decimal digits in N.

You will get $\max\left(0, 25 - \left\lceil \frac{|D-X|}{10} \right\rceil\right)$ points, where X is the true answer.

Answer: 2372

Solution: Let's just consider the first constraint: if we were to take a random permutation, the probability it satisfies $\pi(2i) > \pi(2i-1)$ is $\frac{1}{2^{500}}$. Hence, this is an upper bound on the answer.

Furthermore, if we were to consider the $\pi(2i) > \pi(2i+1)$ constraint as being independent from this first constraint, we would have that around $\frac{1000!}{2^{999}}$ permutations satisfy the constraints. This is then a lower bound on the answer (since the two are positively correlated, the first constraint being satisfied increases the probability this second constraint is).

So, we might guess that something like $\frac{1000!}{2^{750}}$ is a pretty good estimate, where $750 \approx \frac{500+999}{2}$. Since $\log_{10} 2 \approx 0.3$, 2^{750} has around 225 decimal digits. The answer $2568 - 225 = 2343$ gets 19 points.

Here is true code for solving this problem, in Python.

```
n = 1000
s = ("<>" * 500)[:-1]

def toPfx(a):
    for i in range(1, len(a)):
        a[i] += a[i - 1]
    return a

dp = [0, 1]
dp = toPfx(dp)

for i in range(2, n + 1):
    newDp = [0] * (i + 1)
    for j in range(1, i + 1):
```

```
        if s[i - 2] == '<':
            newDp[j] = dp[j - 1]
        else:
            newDp[j] = dp[i - 1] - dp[j - 1]
    dp = newDp
    dp = toPfx(dp)

print(dp[n])
```

26. A year is said to be *interesting* if it is the product of 3, not necessarily distinct, primes (for example $2^2 \cdot 5$ is interesting, but $2^2 \cdot 3 \cdot 5$ is not). How many interesting years are there between 5000 and 10000, inclusive?

    For an estimate of E, you will get $\max\left(0, 25 - \left\lceil \frac{|E-X|}{10} \right\rceil\right)$ points, where X is the true answer.

    Answer: 1296

    Solution: The sequence of triprimes (numbers that are the product of three primes) is listed here: `https://oeis.org/A014612`. Denote the nth triprime as a_n. Then, as $n \to \infty$, we have $a_n \sim 2n \ln n/(\ln \ln n)^2$. The value of n that gives $a_n \approx 10000$ is about 2704 and the value of n that gives $a_n \approx 5000$ is about 1353. The approximate number of values between 5000 and 10000 is then $2704 - 1353 = 1351$, which gets 16 points.

    Python code for the exact answer:

```
from primePy import primes

primes_list = primes.upto(10000 / (2 ** 2))

num = 0
for i in range(len(primes_list)):
  for j in range(i, len(primes_list)):
    for k in range(j, len(primes_list)):
      if 5000 <= primes_list[i] * primes_list[j] * primes_list[k] <= 10000:
        num += 1

print(num)
```

27. Sam chooses 1000 random lattice points (x, y) with $1 \leq x, y \leq 1000$ such that all pairs (x, y) are distinct. Let N be the expected size of the maximum collinear set among them. Estimate $\lfloor 100N \rfloor$. Let S be the answer you provide and X be the true value of $\lfloor 100N \rfloor$.

    You will get $\max\left(0, 25 - \left\lceil \frac{|S-X|}{10} \right\rceil\right)$ points for your estimate.

    Answer: 602

    Solution: Here is C++ code.

```
#include <bits/stdc++.h>

using namespace std;
using ll = long long;
```

```
using ld = long double;

struct Fraction {
    ll numerator, denominator;
    bool neg;

    Fraction(ll num, ll denom) {
        if (num == 0) {
            this->numerator = 0;
            this->denominator = 1;
            this->neg = false;
            return;
        } else if (denom == 0) {
            this->numerator = 1;
            this->denominator = 0;
            this->neg = false;
            return;
        }
        auto d = gcd(abs(num), abs(denom));
        this->numerator = abs(num / d);
        this->denominator = abs(denom / d);
        this->neg = (num < 0) ^ (denom < 0);
    }

    Fraction(ll num) {
        this->neg = num < 0;
        this->numerator = num;
        this->denominator = 1;
    }

    bool operator==(Fraction const& f) const {
        return (neg == f.neg) && numerator * f.denominator == denominator * f.numerator;
    }

    bool operator<(Fraction const& f) const {
        if (neg && !f.neg) return true;
        else if (!neg && f.neg) return false;
        else if (neg && f.neg)
            return -numerator * f.denominator < -denominator * f.numerator;
        else return numerator * f.denominator < denominator * f.numerator;
    }

    friend void operator*=(Fraction& f1, Fraction& f2) {
        ll num1 = f1.numerator * (f1.neg ? -1 : 1);
        ll num2 = f2.numerator * (f2.neg ? -1 : 1);
        f1 = Fraction{num1 * num2, f1.denominator * f2.denominator};
    }
```

```
    friend void operator/=(Fraction& f1, Fraction& f2) {
        ll num1 = f1.numerator * (f1.neg ? -1 : 1);
        ll num2 = f2.numerator * (f2.neg ? -1 : 1);
        f1 = Fraction{num1 * f2.denominator, f1.denominator * num2};
    }

    friend void operator+=(Fraction& f1, Fraction& f2) {
        ll num1 = f1.numerator * (f1.neg ? -1 : 1);
        ll num2 = f2.numerator * (f2.neg ? -1 : 1);
        f1 = Fraction{num1 * f2.denominator + num2 * f1.denominator,
                        f1.denominator * f2.denominator};
    }

    friend void operator-=(Fraction& f1, Fraction& f2) {
        ll num1 = f1.numerator * (f1.neg ? -1 : 1);
        ll num2 = f2.numerator * (f2.neg ? -1 : 1);
        f1 = Fraction{num1 * f2.denominator - num2 * f1.denominator,
                        f1.denominator * f2.denominator};
    }

    friend ostream& operator<<(ostream& os, const Fraction& f) {
        return os << (f.neg ? "-" : "+") << f.numerator << "/" << f.denominator;
    }
};

const Fraction INF = {1, 0};
const int TRIES = 1e4;
const int SIZE = 1000;

ll solve(vector<pair<int, int>>& c) {
    int n = c.size();

    vector<pair<Fraction, Fraction>> data;
    for (int i = 0; i < n; i++) {
        for (int j = i + 1; j < n; j++) {
            auto [x1, y1] = c[i];
            auto [x2, y2] = c[j];
            Fraction slope = {y2 - y1, x2 - x1};
            Fraction intercept = {y1 * x2 - y2 * x1, x2 - x1};
            if (slope == INF) {
                intercept = x1; // take x intercept
            }
            data.push_back({slope, intercept});
        }
    }
    sort(data.begin(), data.end());

    pair<Fraction, Fraction> curr = data[0];
```

```
    int currLength = 1;
    int maxLength = 1;
    for (int i = 1; i < data.size(); i++) {
        if (data[i] == curr) {
            ++currLength;
        } else {
            curr = data[i];
            currLength = 1;
        }
        maxLength = max(maxLength, currLength);
    }

    for (int i = 1; i <= 3 * maxLength; i++) {
        if (i * (i - 1) / 2 == maxLength) {
            return i;
        }
    }
    cout << maxLength << "\n";
    assert(false);
}

int main() {
    vector<pair<int, int>> coords(SIZE * SIZE);
    for (int i = 0; i < SIZE; i++) {
        for (int j = 0; j < SIZE; j++) {
            coords[i * SIZE + j] = {i, j};
        }
    }

    std::default_random_engine rng{};
    vector<ll> results(TRIES);
    for (int t = 0; t < TRIES; t++) {
        cout << "Try " << t << "\n";
        shuffle(coords.begin(), coords.end(), rng);

        vector<pair<int, int>> selected = coords;
        selected.resize(SIZE);
        results[t] = solve(selected);
    }

    ld avg = 0;
    ld sq = 0;
    for (int t = 0; t < TRIES; t++) {
        avg += results[t];
        sq += results[t] * results[t];
    }

    cout << "AVERAGE: " << fixed << setprecision(12) << avg / TRIES << "\n";
```

```
    cout << "STDEV: " << sqrtl(sq / TRIES - (avg / TRIES) * (avg / TRIES)) << "\n";
}
```