1. Let ω be a circle with radius 1. Equilateral triangle $\triangle ABC$ is tangent to ω at the midpoint of side BC and ω lies outside $\triangle ABC$. If line AB is tangent to ω, compute the side length of $\triangle ABC$.

   Answer: $\mathbf{\frac{2\sqrt{3}}{3}}$

   Solution: Let the center of ω be point O and let line AB be tangent to ω at point D. We see that $\triangle ADO$ is a 30-60-90 triangle with $OD = 1$, so $OA = 2$. Then, the height of $\triangle ABC$ is 1. We can then compute that half the side length of $\triangle ABC$ is $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$, so the side length of $\triangle ABC$ is $\boxed{\frac{2\sqrt{3}}{3}}$.

2. Triangle $\triangle ABC$ has side lengths $AB = 3, AC = 2$ and angle $\angle CBA = 30^\circ$. Let the possible lengths of BC be l_1 and l_2, where $l_2 > l_1$. Compute $\frac{l_2}{l_1}$.

   Answer: $\mathbf{\frac{17+3\sqrt{21}}{10}}$

   Solution: Let $\angle BCA = \theta$. By the Law of Sines, we have $\frac{\sin\theta}{\sin\angle CBA} = \frac{AB}{AC}$, which gives $\sin\theta = \sin(30^\circ)\cdot\frac{3}{2} = \frac{3}{4}$. Let the locations of point C corresponding to l_1 and l_2 be C_1 and C_2, and the resulting measures of $\angle BCA$ be θ_1 and θ_2. Then, we have $l_2/l_1 = \frac{\sin(180^\circ-\theta_1-30^\circ)}{\sin(180^\circ-\theta_2-30^\circ)} = \frac{\sin(\theta_1+30^\circ)}{\sin(\theta_1-30^\circ)}$. Since $\theta_2 = 180^\circ - \theta_1$, we get $l_2/l_1 = \frac{\sin(\theta_1+30^\circ)}{\sin(\theta_1-30^\circ)}$. We know that $\sin\theta_1 = \frac{3}{4}$, and since θ_1 corresponds to the shorter length of BC, we have $\cos\theta_1 = \sqrt{1-(3/4)^2} = \frac{\sqrt{7}}{4}$. Then,

$$\begin{aligned} l_2/l_1 &= \frac{\sin(\theta_1+30^\circ)}{\sin(\theta_1-30^\circ)} \\ &= \frac{\sin\theta_1\cos 30^\circ + \cos\theta_1\sin 30^\circ}{\sin\theta_1\cos 30^\circ - \cos\theta_1\sin 30^\circ} \\ &= \frac{(3/4)(\sqrt{3}/2) + (\sqrt{7}/4)(1/2)}{(3/4)(\sqrt{3}/2) - (\sqrt{7}/4)(1/2)} \\ &= \boxed{\frac{17+3\sqrt{21}}{10}}. \end{aligned}$$

3. Triangle $\triangle ABC$ has side lengths $AB = 5, BC = 8$, and $CA = 7$. Let the perpendicular bisector of BC intersect the circumcircle of $\triangle ABC$ at point D on minor arc BC and point E on minor arc AC, and AC at point F. The line parallel to BC passing through F intersects AD at point G and CE at point H. Compute $\frac{[CHF]}{[DGF]}$. (Given a triangle $\triangle ABC$, $[ABC]$ denotes its area.)

   Answer: $\mathbf{\frac{10}{21}}$

   Solution: Note that F is the midpoint of the chord of (ABC) passing through F and parallel to BC. By the Butterfly Theorem, F is also the midpoint of GH. Then, $\frac{[CHF]}{[DGF]}$ is equal to the ratio of the heights of $\triangle CHF$ and $\triangle DGF$. Let the midpoint of BC be M. The height of $\triangle CHF$ is $FM = \frac{1}{2}BC\tan\angle C$ and the height of $\triangle DGF$ is $FD = FM + MD = \frac{1}{2}BC\tan\angle C + \frac{1}{2}BC\tan\angle A/2$ since $\angle MCD = \angle BCD = \angle A/2$.

   Using the Law of Cosines on $\triangle ABC$, we have $\cos\angle A = \frac{5^2+7^2-8^2}{2\cdot5\cdot7} = \frac{1}{7}$ and $\cos\angle C = \frac{7^2+8^2-5^2}{2\cdot7\cdot8} = \frac{11}{14}$. Then, $\tan\angle A/2 = \sqrt{\frac{1-1/7}{1+1/7}} = \frac{\sqrt{3}}{2}$ and $\tan\angle C = \frac{5\sqrt{3}}{11}$.

   Our answer is $\frac{FM}{FM+MD} = \frac{5\sqrt{3}/11}{5\sqrt{3}/11+\sqrt{3}/2} = \boxed{\frac{10}{21}}$.