1. Let $A_1A_2\dots A_{12}$ be a regular dodecagon. Equilateral triangles $\triangle A_1A_2B_1, \triangle A_2A_3B_2, \dots,$ and $\triangle A_{12}A_1B_{12}$ are drawn such that points $B_1, B_2, \dots,$ and B_{12} lie outside dodecagon $A_1A_2\dots A_{12}$. Then, equilateral triangles $\triangle A_1A_2C_1, \triangle A_2A_3C_2, \dots,$ and $\triangle A_{12}A_1C_{12}$ are drawn such that points $C_1, C_2, \dots,$ and C_{12} lie inside dodecagon $A_1A_2\dots A_{12}$. Compute the ratio of the area of dodecagon $B_1B_2\dots B_{12}$ to the area of dodecagon $C_1C_2\dots C_{12}$.

   Answer: $\mathbf{4+2\sqrt{3}}$

   Solution: Each interior angle of a regular dodecagon has measure $\frac{10\cdot 180^\circ}{12} = 150^\circ$. Suppose the side length of $A_1A_2\dots A_{12}$ is 1.

   We can compute the side length of $B_1B_2\dots B_{12}$ by finding B_1B_2. We have $\angle B_1A_2B_2 = 360^\circ - 150^\circ - 60^\circ - 60^\circ = 90^\circ$. Then, $\triangle B_1A_2B_2$ is a 45-45-90 triangle, which gives us $B_1B_2 = \sqrt{2}$.

   We can compute the side length of $C_1C_2\dots C_{12}$ by finding C_1C_2. We have $\angle C_1A_2C_2 = 150^\circ - 60^\circ - 60^\circ = 30^\circ$. Then, using the Law of Cosines in $\triangle C_1A_2C_2$ gives us $C_1C_2 = \sqrt{1+1-2\cos(30^\circ)} = \frac{\sqrt{3}-1}{\sqrt{2}}$.

   The ratio of the area of $B_1B_2\dots B_{12}$ to the area of $C_1C_2\dots C_{12}$ is then $\left(\frac{\sqrt{2}}{\frac{\sqrt{3}-1}{\sqrt{2}}}\right)^2 = \boxed{4+2\sqrt{3}}$.

2. Triangle $\triangle ABC$ has side lengths $AB = 39$, $BC = 16$, and $CA = 25$. What is the volume of the solid formed by rotating $\triangle ABC$ about line BC?

   Answer: $\mathbf{1200\pi}$

   Solution: Let the foot of the perpendicular from A to line BC be D. The volume we want to find can be calculated by subtracting the volume of the cone formed by rotating $\triangle ACD$ from the cone formed by rotating $\triangle ABD$. Let $CD = x$ and $AD = y$. By the Pythagorean theorem, we have

   $$(x+16)^2 + y^2 = 39^2$$

   and

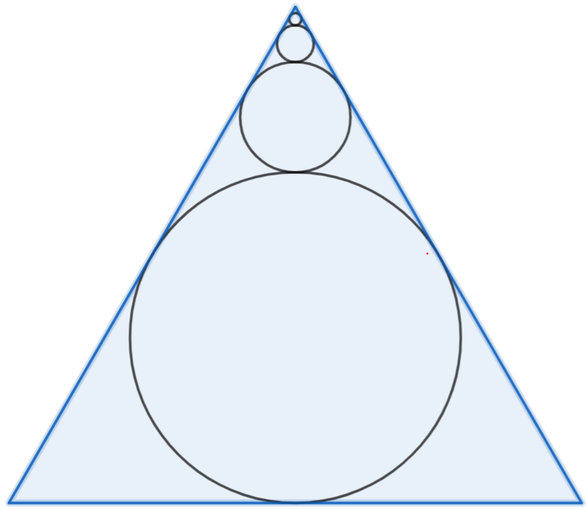
   $$x^2 + y^2 = 25^2.$$

   Subtracting the second equation from the first and solving for x gives $x = \frac{39^2-25^2-16^2}{2\cdot 16} = 20$. Then, $y = \sqrt{25^2-20^2} = 15$. Then, the volume we want is $\frac{1}{3}(15^2\pi)(BD-CD) = \frac{1}{3}(15^2\pi)(16) = \boxed{1200\pi}$.

3. Consider an equilateral triangle $\triangle ABC$ of side length 4. In the zeroth iteration, draw a circle Ω_0 tangent to all three sides of the triangle. In the first iteration, draw circles $\Omega_{1A}, \Omega_{1B}, \Omega_{1C}$ such that circle Ω_{1v} is externally tangent to Ω_0 and tangent to the two sides that meet at vertex v (for example, Ω_{1A} would be tangent to Ω_0 and sides AB, AC). In the nth iteration, draw circle Ω_{nv} externally tangent to $\Omega_{n-1,v}$ and the two sides that meet at vertex v. Compute the total area of all the drawn circles as the number of iterations approaches infinity.

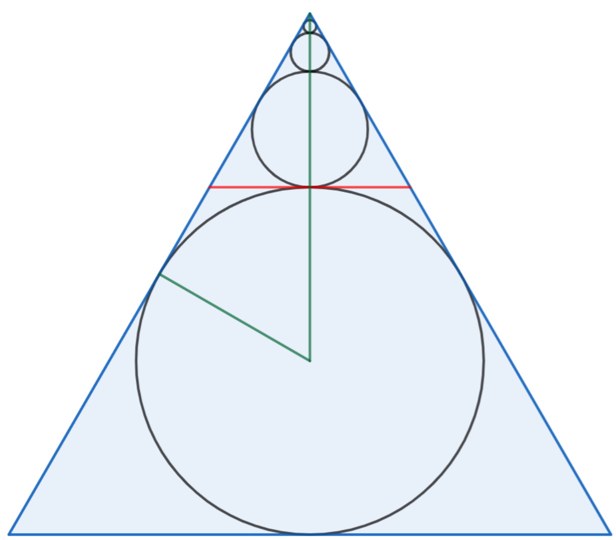
   Answer: $\mathbf{\frac{11\pi}{6}}$

   Solution: Instead of considering all the circles at once, we start by only worrying about circles that are tangent to sides AB and AC, so this would be $\Omega_0, \Omega_{1A}, \Omega_{2A}, ..., \Omega_{iA}, ...$ Note that by symmetry, if we can find the total area of these circles, we can simply multiply by three and then subtract by twice the area of Ω_0 (it's counted three times if we just multiply by 3) to get the desired answer. The figure under consideration is:



The radius of Ω_0 is $\frac{2\sqrt{3}}{3}$ since connecting the center of Ω_0 to the midpoint of any side of the equilateral triangle forms a 30-60-90 triangle (green) with a longer leg of length $4/2 = 2$. Next, we can draw the line tangent to Ω_0 and Ω_{1A} at their intersection (red) to form a smaller version of the exact same shape. The ratio between these two iterations can be found by taking the ratio of the heights of their circumscribed triangles, and is

$$\frac{2\sqrt{3} - \frac{4\sqrt{3}}{3}}{2\sqrt{3}} = \frac{1}{3}.$$



Sol one: Algebra

Note that taking away Ω_0 from the area of all the circles is exactly the same as if we scaled down the area by $\left(\frac{1}{3}\right)^2$. Letting πA be the total area, we then have the relation

$$\pi A - \frac{4\pi}{3} = \frac{\pi}{9} A.$$

Solving this gives

$$A = \frac{3}{2}.$$

The final area is

$$3 \cdot \frac{3\pi}{2} - \frac{8\pi}{3} = \boxed{\frac{11\pi}{6}}.$$

Sol two: Direct calculation

The area of Ω_0 is $\frac{4\pi}{3}$. The sum of the areas of the circles tangent to sides 1 and 2, minus the area of Ω_0, can be computed as $\sum_{i=1}^{\infty} \left(\frac{1}{9}\right)^i \frac{4\pi}{3} = \frac{4\pi}{3} \sum_{i=1}^{\infty} \left(\frac{1}{9}\right)^i = \frac{4\pi}{3} \cdot \frac{1}{8} = \frac{\pi}{6}$. The total area is then $3 \cdot \frac{\pi}{6} + \frac{4\pi}{3} = \boxed{\frac{11\pi}{6}}$.

4. Equilateral triangle $\triangle ABC$ is inscribed in circle Ω, which has a radius of 1. Let the midpoint of BC be M. Line AM intersects Ω again at point D. Let ω be the circle with diameter MD. Compute the radius of the circle that is tangent to BC on the same side of BC as ω, internally tangent to Ω, and externally tangent to ω.

   Answer: $\mathbf{\frac{3}{16}}$

   Solution: Denote the circle whose radius we want to compute as ω'. Let the center of Ω be O_1, the center of ω be O_2, and the center of ω' be O_3. Since O_1BM is a 30-60-90 triangle, we have $O_1M = \frac{1}{2}$. Then we see that the diameter of ω is $1 - \frac{1}{2} = \frac{1}{2}$, so the radius of ω is $\frac{1}{4}$. Let the radius of ω' be r. Let the foot of the perpendicular from O_3 to MD be point E. We have by the Pythagorean theorem

$$\begin{aligned} O_3E &= \sqrt{O_3O_2^2 - O_2E^2} \\ &= \sqrt{\left(\frac{1}{4} + r\right)^2 - \left(\frac{1}{4} - r\right)^2} \\ &= \sqrt{r}. \end{aligned}$$

   Then, by the Pythagorean theorem in $\triangle O_1O_3E$ we have

$$O_3E^2 + O_1E^2 = O_1O_3^2,$$

   which becomes

$$(\sqrt{r})^2 + \left(\frac{1}{2} + r\right)^2 = (1 - r)^2.$$

   Solving for r gives us $r = \boxed{\frac{3}{16}}$.

5. Equilateral triangle $\triangle ABC$ has side length 12 and equilateral triangles of side lengths $a, b, c < 6$ are each cut from a vertex of $\triangle ABC$, leaving behind an equiangular hexagon $A_1A_2B_1B_2C_1C_2$, where A_1 lies on AC, A_2 lies on AB, and the rest of the vertices are similarly defined. Let A_3 be the midpoint of A_1A_2 and define B_3, C_3 similarly. Let the center of $\triangle ABC$ be O. Note that OA_3, OB_3, OC_3 split the hexagon into three pentagons. If the sum of the areas of the equilateral triangles cut out is $18\sqrt{3}$ and the ratio of the areas of the pentagons is $5 : 6 : 7$, what is the value of abc?

Answer: $64\sqrt{3}$

Solution: For the pentagon determined by OA_3 and OB_3, we can split it into $\triangle OA_3A_2, \triangle OA_2B_1, \triangle OB_1B_3$ and determine its area as

$$\frac{1}{2}\cdot\frac{a}{2}\cdot\left(4\sqrt{3}-\frac{a\sqrt{3}}{2}\right)+\frac{1}{2}\cdot 2\sqrt{3}\cdot(12-a-b)+\frac{1}{2}\cdot\frac{b}{2}\cdot\left(4\sqrt{3}-\frac{b\sqrt{3}}{2}\right)=\frac{\sqrt{3}}{8}(96-a^2-b^2).$$

Similarly, the areas of the other pentagons are $\frac{\sqrt{3}}{8}(96-b^2-c^2)$ and $\frac{\sqrt{3}}{8}(96-c^2-a^2)$. From the fact that the sum of the areas of the equilateral triangles cut out is $18\sqrt{3}$, we find that $a^2+b^2+c^2=\frac{4}{\sqrt{3}}\cdot 18\sqrt{3}=72$. Then,

$$96-a^2-b^2+96-b^2-c^2+96-c^2-a^2=288-2(a^2+b^2+c^2)=288-144=144.$$

Dividing 144 into the ratio $5:6:7$ gives $40, 48, 56$ as values for $96-a^2-b^2, 96-b^2-c^2, 96-c^2-a^2$. Then, we have $56, 48, 40$ for the values of $a^2+b^2, b^2+c^2, c^2+a^2$, which we can use along with $a^2+b^2+c^2=72$ to find that a, b, c are $4, 2\sqrt{6}, 4\sqrt{2}$ (in any order). Then, $abc=4\cdot 2\sqrt{6}\cdot 4\sqrt{2}=\boxed{64\sqrt{3}}$.

6. Let ABC be a triangle and ω_1 its incircle. Let points D and E be on segments AB, AC respectively such that DE is parallel to BC and tangent to ω_1. Now let ω_2 be the incircle of $\triangle ADE$ and let points F and G be on segments AD, AE respectively such that FG is parallel to DE and tangent to ω_2. Given that ω_2 is tangent to line AF at point X and line AG at point Y, the radius of ω_1 is 60, and

$$4(AX)=5(FG)=4(AY),$$

compute the radius of ω_2.

Answer: 12

Solution: Let s be the semiperimeter of $\triangle AFG$, r the inradius of $\triangle AFG$, and a the length of FG. It is well-known that $AX=AY=s$ since ω_2 is an excircle of $\triangle AFG$. Then, if we let $s=ka$, from the problem statement we have $k=\frac{5}{4}$. The radius of ω_2 is $\frac{[AFG]}{s-a}$ using the formula for the radius of an excircle. We have

$$\begin{aligned}\frac{[AFG]}{s-a}&=\frac{rs}{s-a}\\&=\frac{rka}{ka-a}\\&=\left(\frac{k}{k-1}\right)r.\end{aligned}$$

Since ω_2 is the incircle of $\triangle ADE$, we see that $\triangle AFG\sim\triangle ADE$ with ratio $\frac{k}{k-1}$. A similar calculation gives us that the radius of ω_1 is $\left(\frac{k}{k-1}\right)^2 r$. Then, the answer is

$$\frac{60}{\frac{k}{k-1}}=\frac{60}{5}=\boxed{12}.$$

7. Triangle ABC has $AC=5$. D and E are on side BC such that AD and AE trisect $\angle BAC$, with D closer to B and $DE=\frac{3}{2}$, $EC=\frac{5}{2}$. From B and E, altitudes BF and EG are drawn onto side AC. Compute $\frac{CF}{CG}-\frac{AF}{AG}$.

Answer: 2

Solution: Let $\angle BAC = 3a$ and $AB = x$. Observe first that, by the angle bisector theorem, $\frac{AD}{AC} = \frac{DE}{EC}$, so $AD = AC \cdot \frac{DE}{EC} = 5 \cdot \frac{3}{5} = 3$. Then, $\triangle ADC$ is a 3-4-5 triangle, so $\angle ADC = 90°$. Since AD bisects $\angle BAE$ and is perpendicular to BE, we have that $\triangle BAE$ is isosceles, which gives us $BD = \frac{3}{2}$ and $AE = AB = x$.

Now we know that $BF = x\sin(3a)$, $AF = x\cos(3a)$, $EG = x\sin(a)$, and $AG = x\cos(a)$. Since BF is parallel to EG, $\frac{CF}{CG} = \frac{BF}{EG} = \frac{x\sin(3a)}{x\sin(a)} = 3 - 4\sin^2 a$, using the triple angle formula. Similarly, $\frac{AF}{AG} = \frac{x\cos(3a)}{x\cos(a)} = 4\cos^2 a - 3$.

Then $\frac{CF}{CG} - \frac{AF}{AG} = 3 - 4\sin^2 a - 4\cos^2 a + 3 = 6 - 4 = \boxed{2}$.

8. In triangle $\triangle ABC$, point R lies on the perpendicular bisector of AC such that CA bisects $\angle BAR$. Line BR intersects AC at Q, and the circumcircle of $\triangle ARC$ intersects segment AB at $P \neq A$. If $AP = 1, PB = 5$, and $AQ = 2$, compute AR.

Answer: $\mathbf{\frac{3+3\sqrt{129}}{16}}$

Solution: Since R lies on the perpendicular bisector,

$$\angle ACR = \angle RAC = \angle CAB$$

and AB is parallel to RC. Thus, $\triangle AQB \sim \triangle CQR$ and

$$QC = \frac{RC}{AB} \cdot AQ = \frac{RC}{3}.$$

Let $AR = RC = x$. Then as AB is parallel to RC and $ARCP$ is cyclic,

$$PC = AR = x$$

and

$$PR = AC = 2 + \frac{x}{3}.$$

It follows from Ptolemy's theorem that

$$(AP)(RC) + (AR)(PC) = (AC)(PR),$$

which gives us

$$\left(2 + \frac{x}{3}\right)^2 = x + x^2$$

which comes out to

$$8x^2 - 3x - 36 = 0.$$

Using the quadratic formula, the only positive solution is

$$x = \boxed{\frac{3 + 3\sqrt{129}}{16}}.$$

9. Triangle $\triangle ABC$ is isosceles with $AC = AB$, $BC = 1$, and $\angle BAC = 36°$. Let ω be a circle with center B and radius $r_\omega = \frac{P_{ABC}}{4}$, where P_{ABC} denotes the perimeter of $\triangle ABC$. Let ω intersect line AB at P and line BC at Q. Let I_B be the center of the excircle with of $\triangle ABC$

with respect to point B, and let BI_B intersect PQ at S. We draw a tangent line from S to $\odot I_B$ that intersects $\odot I_B$ at point T. Compute the length of ST.

Answer: $\frac{7+3\sqrt{5}}{16}$

Solution: First note that if the excircle touches lines BA, BC at points E, F respectively, then $BE = BF = \frac{P_{ABC}}{2}$, and, hence, P and Q are midpoints. This means that P and Q have equal power with respect to $\odot I_B$ and B (if we consider B to be a circle with zero radius). We therefore get that PQ is the radical axis of $\odot I_B$ and B, and every point on PQ has equal power with respect to $\odot I_B$ and B. This implies that $ST = SB$, and it is therefore sufficient to find SB.

We now need to find P_{ABC}. We recall that $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$, and hence

$$AC = \frac{1}{2\sin 18^\circ} = \frac{2}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{2}.$$

This gives us $P_{ABC} = \sqrt{5}+2$, and by extension $PB = \frac{P_{ABC}}{4} = \frac{\sqrt{5}+2}{4}$. We now recall $\sin 54^\circ = \frac{\sqrt{5}+1}{4}$, and this gives us the final answer

$$SB = \frac{\sqrt{5}+2}{4} \cdot \frac{\sqrt{5}+1}{4} = \boxed{\frac{7+3\sqrt{5}}{16}} = ST.$$

10. Let $\triangle ABC$ be a triangle with side lengths $AB = 13, BC = 14$, and $CA = 15$. The angle bisector of $\angle BAC$, the angle bisector of $\angle ABC$, and the angle bisector of $\angle ACB$ intersect the circumcircle of $\triangle ABC$ again at points D, E and F, respectively. Compute the area of hexagon $AFBDCE$.

Answer: $\frac{1365}{8}$

Solution: Let the incenter of $\triangle ABC$ be I. By the Incenter/Excenter Lemma, D is the center of (IBC), where (ABC) denotes the circle circumscribing $\triangle ABC$. Similarly, E is the center of (IAC) and F is the center of (IAB). Denote the excenter opposite vertex A as I_A, and define I_B, I_C similarly. Also by the lemma, we can angle chase to find that $\triangle ABC$ is the orthic triangle of $\triangle I_A I_B I_C$.

Note that (ABC) is the nine-point circle of $\triangle I_A I_B I_C$. Let the radius of $(I_A I_B I_C)$ be R' and denote $\angle BAC = \angle A, \angle ABC = \angle B, \angle BCA = \angle C$. By Law of Sines on $\triangle I I_A I_B$, we have $\frac{II_A}{\sin \angle II_B I_A} = \frac{I_A I_B}{\sin \angle I_A I I_B}$, which gives us $\frac{II_A}{\sin(A/2)} = \frac{I_A I_B}{\sin(180^\circ - \angle I_B I_C I_A)} = \frac{I_A I_B}{\sin \angle I_B I_C I_A} = 2R'$. Then, $II_A = 2R' \sin(A/2)$. Now, note that there is a homothety from $(I_A I_B I_C)$ to its nine-point circle. Let the radius of (ABC) be R. Then, $II_A = 4R\sin(A/2)$. Similarly, $II_B = 4R\sin(B/2)$ and $II_C = 4R\sin(C/2)$.

Then, the area of $AFBDCE$ is

$$\begin{aligned}[ABC] + [DBC] + [EAC] + [FAB] &= 84 + \frac{1}{2}(II_A/2)^2 \sin(180^\circ - A) \\ &\quad + \frac{1}{2}(II_B/2)^2 \sin(180^\circ - B) + \frac{1}{2}(II_C/2)^2 \sin(180^\circ - C) \\ &= 84 + 2R^2 \sin^2(A/2)\sin A + 2R^2 \sin^2(B/2)\sin B + 2R^2\sin^2(C/2)\sin C \\ &= 84 + R^2(1-\cos A)\sin A + R^2(1-\cos B)\sin B + R^2(1-\cos C)\sin C,\end{aligned}$$

where the area $[ABC]$ can be computed by noting that $\triangle ABC$ can be formed from a 5-12-13 and 9-12-15 triangle. This also allows us to find with Law of Sines that $\sin A = \frac{56}{65}, \cos A =$

$\frac{33}{65}, \sin B = \frac{12}{13}, \cos B = \frac{5}{13}, \sin C = \frac{4}{5}$, and $\cos C = \frac{3}{5}$. Also, $R = \frac{13 \cdot 14 \cdot 15}{4 \cdot 84} = \frac{65}{8}$ using the formula for the length of the circumradius of a triangle. Finally, we have

$$\begin{aligned}[AFBDCE] &= 84 + (65/8)^2 \cdot ((1 - 33/65) \cdot 56/65 + (1 - 3/5) \cdot 4/5 + (1 - 5/13) \cdot 12/13) \\ &= \boxed{\frac{1365}{8}}.\end{aligned}$$