Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise.
No calculators.

1. Let $A_1A_2\dots A_{12}$ be a regular dodecagon. Equilateral triangles $\triangle A_1A_2B_1, \triangle A_2A_3B_2, \dots,$ and $\triangle A_{12}A_1B_{12}$ are drawn such that points $B_1, B_2, \dots,$ and B_{12} lie outside dodecagon $A_1A_2\dots A_{12}$. Then, equilateral triangles $\triangle A_1A_2C_1, \triangle A_2A_3C_2, \dots,$ and $\triangle A_{12}A_1C_{12}$ are drawn such that points $C_1, C_2, \dots,$ and C_{12} lie inside dodecagon $A_1A_2\dots A_{12}$. Compute the ratio of the area of dodecagon $B_1B_2\dots B_{12}$ to the area of dodecagon $C_1C_2\dots C_{12}$.

2. Triangle $\triangle ABC$ has side lengths $AB = 39$, $BC = 16$, and $CA = 25$. What is the volume of the solid formed by rotating $\triangle ABC$ about line BC?

3. Consider an equilateral triangle $\triangle ABC$ of side length 4. In the zeroth iteration, draw a circle Ω_0 tangent to all three sides of the triangle. In the first iteration, draw circles $\Omega_{1A}, \Omega_{1B}, \Omega_{1C}$ such that circle Ω_{1v} is externally tangent to Ω_0 and tangent to the two sides that meet at vertex v (for example, Ω_{1A} would be tangent to Ω_0 and sides AB, AC). In the nth iteration, draw circle Ω_{nv} externally tangent to $\Omega_{n-1,v}$ and the two sides that meet at vertex v. Compute the total area of all the drawn circles as the number of iterations approaches infinity.

4. Equilateral triangle $\triangle ABC$ is inscribed in circle Ω, which has a radius of 1. Let the midpoint of BC be M. Line AM intersects Ω again at point D. Let ω be the circle with diameter MD. Compute the radius of the circle that is tangent to BC on the same side of BC as ω, internally tangent to Ω, and externally tangent to ω.

5. Equilateral triangle $\triangle ABC$ has side length 12 and equilateral triangles of side lengths $a, b, c < 6$ are each cut from a vertex of $\triangle ABC$, leaving behind an equiangular hexagon $A_1A_2B_1B_2C_1C_2$, where A_1 lies on AC, A_2 lies on AB, and the rest of the vertices are similarly defined. Let A_3 be the midpoint of A_1A_2 and define B_3, C_3 similarly. Let the center of $\triangle ABC$ be O. Note that OA_3, OB_3, OC_3 split the hexagon into three pentagons. If the sum of the areas of the equilateral triangles cut out is $18\sqrt{3}$ and the ratio of the areas of the pentagons is $5 : 6 : 7$, what is the value of abc?

6. Let ABC be a triangle and ω_1 its incircle. Let points D and E be on segments AB, AC respectively such that DE is parallel to BC and tangent to ω_1. Now let ω_2 be the incircle of $\triangle ADE$ and let points F and G be on segments AD, AE respectively such that FG is parallel to DE and tangent to ω_2. Given that ω_2 is tangent to line AF at point X and line AG at point Y, the radius of ω_1 is 60, and
$$4(AX) = 5(FG) = 4(AY),$$
compute the radius of ω_2.

7. Triangle ABC has $AC = 5$. D and E are on side BC such that AD and AE trisect $\angle BAC$, with D closer to B and $DE = \frac{3}{2}$, $EC = \frac{5}{2}$. From B and E, altitudes BF and EG are drawn onto side AC. Compute $\frac{CF}{CG} - \frac{AF}{AG}$.

8. In triangle $\triangle ABC$, point R lies on the perpendicular bisector of AC such that CA bisects $\angle BAR$. Line BR intersects AC at Q, and the circumcircle of $\triangle ARC$ intersects segment AB at $P \neq A$. If $AP = 1, PB = 5$, and $AQ = 2$, compute AR.

9. Triangle $\triangle ABC$ is isosceles with $AC = AB$, $BC = 1$, and $\angle BAC = 36^\circ$. Let ω be a circle with center B and radius $r_\omega = \frac{P_{ABC}}{4}$, where P_{ABC} denotes the perimeter of $\triangle ABC$. Let ω intersect line AB at P and line BC at Q. Let I_B be the center of the excircle with of $\triangle ABC$ with respect to point B, and let BI_B intersect PQ at S. We draw a tangent line from S to $\odot I_B$ that intersects $\odot I_B$ at point T. Compute the length of ST.

10. Let $\triangle ABC$ be a triangle with side lengths $AB = 13, BC = 14$, and $CA = 15$. The angle bisector of $\angle BAC$, the angle bisector of $\angle ABC$, and the angle bisector of $\angle ACB$ intersect the circumcircle of $\triangle ABC$ again at points D, E and F, respectively. Compute the area of hexagon $AFBDCE$.