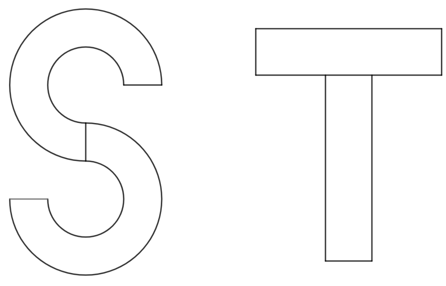
1. Arpit begins with a circular paper disk of radius 8 and cuts out a circular disk of radius 4 with the same center, which results in an annulus. Then, he cuts out and removes a sector of the annulus with a central angle of 90°. He repeats these steps beginning with another circular disk of radius 8 and glues together the two annulus sectors in the shape of an S as shown on the left below. After making the S, Arpit also makes a letter T by gluing together two rectangles each having dimensions $x \times 4x$ as shown on the right below. Compute the value of x Arpit should choose so that the S and T have the same area.



Answer: $\mathbf{3\sqrt{\pi}}$

Solution: The area of the original circle is 64π. After cutting out an inner circle of radius 4, the remaining area is $64\pi \cdot \frac{3}{4} = 48\pi$. After cutting out a sector with central angle 90°, the remaining area is $48\pi \cdot \frac{3}{4} = 36\pi$. The area of the S is $36\pi \cdot 2 = 72\pi$. Because the T has two rectangles, each rectangle has area $\frac{72\pi}{2} = 36\pi$. We then have $x \cdot 4x = 36\pi$, so $x = \boxed{3\sqrt{\pi}}$.

2. Charlie has a large supply of chocolates, each of which has two properties: the filling can be raspberry or blueberry and the shape can be circular or square. Half of the chocolates have a raspberry filling and the other half have a blueberry filling. Also, $\frac{4}{9}$ of the raspberry-filled chocolates are circular and $\frac{2}{3}$ of the blueberry-filled chocolates are square. What fraction of the square chocolates are raspberry-filled?

Answer: $\mathbf{\frac{5}{11}}$

Solution 1: We denote the values of each property: R for raspberry-filled, B for blueberry-filled, C for circular, and S for square. We want to find $P(R \mid S)$, the probability that a chocolate is raspberry-filled given that it is square. We have

$$\begin{aligned}
P(R \mid S) &= \frac{P(R,S)}{P(S)} && \text{Definition of conditional probability} \\
&= \frac{P(R,S)}{P(R,S)+P(B,S)} && \text{Law of total probability} \\
&= \frac{P(S \mid R)P(R)}{P(S \mid R)P(R)+P(S \mid B)P(B)} && \text{Chain rule} \\
&= \frac{(1-P(C \mid R))P(R)}{(1-P(C \mid R))P(R)+P(S \mid B)P(B)} && \text{Complement rule} \\
&= \frac{\left(1-\frac{4}{9}\right)\left(\frac{1}{2}\right)}{\left(1-\frac{4}{9}\right)\left(\frac{1}{2}\right)+\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)} \\
&= \boxed{\frac{5}{11}}.
\end{aligned}$$

Solution 2: Let us assume that Charlie has 18 chocolates. We can deduce that there are 9 raspberry-filled chocolates and 9 blueberry-filled chocolates. From this, we can determine that

- 4 chocolates are raspberry-filled and circular
- 5 chocolates are raspberry-filled and square
- 6 chocolates are blueberry-filled and square
- 3 chocolates are blueberry-filled and circular

Hence, there are a total of 11 chocolates that are square of which 5 are raspberry-filled. We determine the fraction of square chocolates that are raspberry-filled are $\boxed{\frac{5}{11}}$.

3. Let p be an odd prime, and P be the *second* smallest multiple of p that is a perfect cube. How many positive factors does P have?

   Answer: 16

   Solution: Since p is an odd prime, the second smallest multiple of p that is a perfect cube is $2^3 \cdot p^3 = 8p^3$. The number of positive factors of $8p^3$ is $(3+1)(3+1) = \boxed{16}$.

4. There exists a unique real value of x such that

$$(x + \sqrt{x})^2 = 16.$$

   Compute x.

   Answer: $\mathbf{\frac{9-\sqrt{17}}{2}}$

   Solution: In order for $\sqrt{x}$ to be defined, $x \geq 0$. Then $x + \sqrt{x} \geq 0$ and $x + \sqrt{x} = 4$. Letting $y = \sqrt{x}$, we get $y^2 + y - 4 = 0$ which by the quadratic formula has solutions $\frac{-1\pm\sqrt{17}}{2}$. As $y = \sqrt{x} \geq 0$, it follows that $y = \frac{-1+\sqrt{17}}{2}$ and

$$x = y^2 = \frac{18 - 2\sqrt{17}}{4} = \boxed{\frac{9-\sqrt{17}}{2}}.$$

5. For all positive integers $n > 1$, let $f(n)$ denote the largest odd proper divisor of n (a proper divisor of n is a positive divisor of n except for n itself). Given that $N = 20^{23} \cdot 23^{20}$, compute

$$\frac{f(N)}{f(f(f(N)))}.$$

   Answer: 25

   Solution: Let $n > 1$ be a positive integer. If n is even, note that $f(n) = \frac{n}{2^{v(n)}}$, where $v(n)$ is the largest integer k such that 2^k divides n. Otherwise, if $n > 1$ is odd, we have $f(n) = \frac{n}{p(n)}$, where $p(n)$ is the smallest odd prime factor of n (which exists since $n > 1$ and n is odd). Using these observations, we find that $f(N) = 5^{23} \cdot 23^{20}$, $f(f(N)) = 5^{22} \cdot 23^{20}$, and $f(f(f(N))) = 5^{21} \cdot 23^{20}$. Our answer is

$$\frac{5^{23} \cdot 23^{20}}{5^{21} \cdot 23^{20}} = \boxed{25}.$$

6. Let X be the set of natural numbers with 10 digits comprising of only 0's and 1's, and whose first digit is 1. How many numbers in X are divisible by 3?

   Answer: 171

   Solution: Let's choose a number $x \in X$. In order for x to be divisible by 3, the sum of the digits of x must be divisible by 3. Hence, x's digits must have a sum of either 3, 6, or 9 since the length of x is 10. Equivalently, this indicates that there are either 3, 6, or 9 amount of 1's in x. Since the first digit is 1, we can choose where to place the other 1's in the remaining 9 places. Thus, the amount of numbers in X that are divisible by 3 is $\binom{9}{2} + \binom{9}{5} + \binom{9}{8} = \boxed{171}$.

7. If x and y are positive integers that satisfy $43x + 47y = 2023$, compute the minimum possible value of $x + y$.

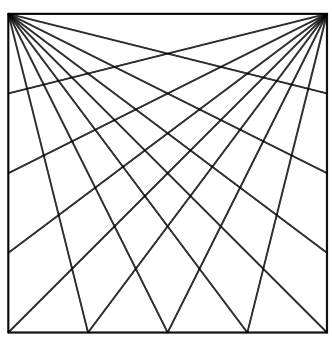
   Answer: 45

   Solution: Taking modulo 43, one notices that $47 \equiv 4 \pmod{43}$ and $2023 \equiv 2 \pmod{43}$. We rewrite $43x + 47y \equiv 4y \equiv 2 \pmod{43}$. The least y for which this holds is $y = 22$, and computing, $23 \cdot 43 + 22 \cdot 47 = 2023$. It is impossible to have other positive integer pair solutions because all solutions are given by the form $(x, y) = (23 + 47k, 22 - 43k)$ for integers k. Therefore, the minimal value is $23 + 22 = \boxed{45}$.

8. Lines are drawn from a corner of a square to partition the square into 8 parts with equal areas. Another set of lines is drawn in the same way from an adjacent corner. How many regions are formed inside the square and are bounded by drawn lines and edges of the square?

   Answer: 58

   Solution: Initially, the lines from the first corner partition the square into 8 regions. Then, seven additional lines are drawn from the adjacent corner. Four of these lines intersect with every line from the first corner and each subsequent line below the diagonal cutting the square in half intersects with one less line from the first corner. Intersecting n lines from the first corner produces $n + 1$ additional regions in the square. Our answer is $8 + 4 \cdot 8 + 7 + 6 + 5 = \boxed{58}$.



9. William has a large supply of candy bars and wants to choose one of among three families to give the candy to. Family A has 13 children, family B has 11 children, and family C has 7 children. The children in family C each require an even number of candy bars. If William attempts to distribute the candy bars equally among the children in families A, B, and C, there are 7, 5, and 8 candy bars left over, respectively. What is the least number of candy bars that William could have?

Answer: 1996

Solution: Let x be the number of candy bars that William has. Then, we are given that $x \equiv 7 \equiv -6 \pmod{13}$, $x \equiv 5 \equiv -6 \pmod{11}$, and $x \equiv 8 \equiv -6 \pmod{14}$. The least common multiple of 13, 11, and 14 is $2 \cdot 7 \cdot 11 \cdot 13 = 2002$, so the least possible value of x is $2002 - 6 = \boxed{1996}$.

10. Consider the rectangle with a length of 2 and a width of 1. Pick one of the two diagonals of the rectangle. Observe that this diagonal separates the rectangle into two right-angled triangles, R_1 and R_2. Reflect R_1 in the diagonal to obtain R_1'. Compute the area of the intersection of R_1' and R_2.

Answer: $\mathbf{\frac{5}{8}}$

Solution 1: Let the rectangle be $ABCD$ where $AB = CD = 2$ and $AD = BC = 1$. Note that triangle $\triangle ADC$ is R_1 and reflected across the diagonal to obtain triangle $\triangle AD'C$ and triangle $\triangle ABC$ is R_2. Let the intersection of $D'C$ and BA be E such that the area of intersection between R_1' and R_2 is triangle $\triangle AEC$. We can determine, through the Pythagorean Theorem, that $AC = \sqrt{AB^2 + BC^2} = \sqrt{2^2 + 1^2} = \sqrt{5}$. Let's drop a perpendicular from E to AC such that the height intersections AC at F. Therefore, $AF = FC = \frac{\sqrt{5}}{2}$. Through the common angle $\angle A$ and right angle, we can determine that $\triangle ABC \sim \triangle AFE$. Hence, we can determine that $\frac{AB}{AF} = \frac{BC}{FE}$. We know the length of $AB = 2$, $AF = \frac{\sqrt{5}}{2}$, and $BC = 1$, so we can determine what FE is through $\frac{AB}{AF} = \frac{BC}{FE} \to \frac{2}{\frac{\sqrt{5}}{2}} = \frac{1}{FE}$. Therefore, $FE = \frac{\sqrt{5}}{4}$. We can determine that our desired area is $\frac{1}{2} \cdot AC \cdot FE = \frac{1}{2} \cdot \sqrt{5} \cdot \frac{\sqrt{5}}{4} = \boxed{\frac{5}{8}}$.

Solution 2: Consider the rectangle on the Cartesian plane with vertices (0,0), (0,1), (2,0), (2,1). Without loss of generality, take the diagonal to be the line $y = \frac{1}{2} \cdot x$. The point (0,1) is reflected to $(\frac{4}{5}, -\frac{3}{5})$. Call this point V, and let the point (2,1) be called W. The key observation is that the area of overlap must necessarily be the triangle bounded by the line VW, the x-axis, and the line $y = \frac{1}{2} \cdot x$. We evaluate that the line VW crosses the x-axis at $(\frac{5}{4}, 0)$. If we take the line segment from the origin to $(\frac{5}{4}, 0)$ to be the base of the triangle, we determine the height is 1. Then the area is just $1 \cdot \frac{5}{4} \cdot \frac{1}{2} = \boxed{\frac{5}{8}}$.

11. Nathan has discovered a new way to construct chocolate bars, but it's expensive! He starts with a single 1×1 square of chocolate and then adds more rows and columns from there. If his current bar has dimensions $w \times h$ (w columns and h rows), then it costs w^2 dollars to add another row and h^2 dollars to add another column. What is the minimum cost to get his chocolate bar to size 20×20?

Answer: 5339

Solution: The optimal way to add rows and columns to the 1×1 chocolate to the 20×20 chocolate is to alternate adding rows and columns. (A rough proof of this is below.) If we do this, then the costs are 1^2 for the first row, plus 2^2 for the first column, plus 2^2 for the second row, plus $3^2 + 3^2 + 4^2 + \cdots$. The formula for the overall cost to get to $n \times n$ is $1^2 + 2 \cdot 2^2 + 2 \cdot 3^2 + \cdots + 2 \cdot (n-1)^2 + n^2$. The sum of the first n squares can be calculated as $\frac{n(n+1)(2n+1)}{6}$. Thus, we can simplify our desired sum to $\frac{n(n+1)(2n+1)}{3} - 1 - n^2$. For $n = 20$ this equals $\frac{20 \cdot 21 \cdot 41}{3} - 1 - 20^2 = \boxed{5339}$.

Proof: Assume $w > h$ (more columns than rows). Adding a column and then a row costs $h^2 + (w+1)^2$. Adding a row and then a column costs $w^2 + (h+1)^2$. Since $w > h$, we have

$h^2 + (w+1)^2 = h^2 + w^2 + 2w + 1 > h^2 + w^2 + 2h + 1 = w^2 + (h+1)^2$. Therefore, it's always more optimal to add a row first in this case. We can see that alternating rows and columns is optimal.

12. Let $A_1A_2\ldots A_{12}$ be a regular dodecagon. Equilateral triangles $\triangle A_1A_2B_1, \triangle A_2A_3B_2, \ldots,$ and $\triangle A_{12}A_1B_{12}$ are drawn such that points $B_1, B_2, \ldots,$ and B_{12} lie outside dodecagon $A_1A_2\ldots A_{12}$. Compute the ratio of the area of dodecagon $B_1B_2\ldots B_{12}$ to the area of dodecagon $A_1A_2\ldots A_{12}$.

    Answer: 2

    Solution: Each interior angle of a regular dodecagon measures $\frac{10\cdot 180^\circ}{12} = 150^\circ$. Suppose the side length of $A_1A_2\ldots A_{12}$ is 1. We can compute the side length of $B_1B_2\ldots B_{12}$ by finding B_1B_2. We have $\angle B_1A_2B_2 = 360^\circ - 150^\circ - 60^\circ - 60^\circ = 90^\circ$. Then, $\triangle B_1A_2B_2$ is a 45-45-90 triangle, which gives us $B_1B_2 = \sqrt{2}$. The ratio of the area of $B_1B_2\ldots B_{12}$ to the area of $A_1A_2\ldots A_{12}$ is $(\sqrt{2})^2 = \boxed{2}$.

13. Two frogs jump along a straight line in the same direction, starting at the same place. Every ten seconds, each frog jumps 2, 4, or 6 feet, with each possibility being equally likely. What is the probability that the frogs have traveled the same distance after thirty seconds?

    Answer: $\frac{47}{243}$

    Solution: If the distance traveled is 6, there is exactly 1 possibility—all jumps of 2. If the distance traveled is 8, each frog has 3 possible sequences, permutations of $\{2,2,4\}$, giving $3\times 3 = 9$ possibilities. If the distance traveled is 10, each frog has 6 possible sequences, permutations of $\{2,2,6\}$ and $\{2,4,4\}$, giving $6\times 6 = 36$ possibilities. If the distance traveled is 12, each frog has 7 possible sequences, permutations of $\{2,4,6\}$ and $\{4,4,4\}$, giving $7\times 7 = 49$ possibilities.

    By symmetry, the number of possibilities for a distance of 18 is the same as that for 6, as well as 16 and 8, and 14 and 10.

    So, the total number of possibilities for the frogs to travel the same distance is $1+9+36+49+36+9+1 = 141$. The total number of possible sequences of jumps for the frogs is $(3^2)^3 = 729$. Finally, we have $\frac{141}{729} = \boxed{\dfrac{47}{243}}$.

14. How many trailing zeros does the value

    $$300 \cdot 305 \cdot 310 \cdots 1090 \cdot 1095 \cdot 1100$$

    end with?

    Answer: 161

    Solution: Rewrite the expression as

    $$300 \cdot 305 \cdot 310 \cdots 1090 \cdot 1095 \cdot 1100 = 5^{161} \cdot \frac{220!}{59!}.$$

    By Legendre's formula, there are

    $$\left\lfloor \frac{59}{5} \right\rfloor + \left\lfloor \frac{59}{25} \right\rfloor = 13$$

    factors of 5 in 59! and

    $$\left\lfloor \frac{59}{2} \right\rfloor + \left\lfloor \frac{59}{4} \right\rfloor + \left\lfloor \frac{59}{8} \right\rfloor + \left\lfloor \frac{59}{16} \right\rfloor + \left\lfloor \frac{59}{32} \right\rfloor = 54$$

factors of 2 in 59!. Morever, there are

$$\left\lfloor \frac{220}{5} \right\rfloor + \left\lfloor \frac{220}{25} \right\rfloor + \left\lfloor \frac{220}{125} \right\rfloor = 53$$

factors of 5 in 220! and

$$\left\lfloor \frac{220}{2} \right\rfloor + \left\lfloor \frac{220}{4} \right\rfloor + \left\lfloor \frac{220}{8} \right\rfloor + \left\lfloor \frac{220}{16} \right\rfloor + \left\lfloor \frac{220}{32} \right\rfloor + \left\lfloor \frac{220}{64} \right\rfloor + \left\lfloor \frac{220}{128} \right\rfloor = 215$$

factors of 2 in 220!. Thus there are

$$161 + 53 - 13 = 201$$

factors of 5 in the product and

$$215 - 54 = 161$$

factors of 2 in the product. Since $161 < 201$, it follows the answer is $\boxed{161}$.

15. Let A, B, and C be three points on a line (in that order), and let X and Y be two points on the same side of line AC. If $\triangle AXB \sim \triangle BYC$ and the ratio of the area of quadrilateral $AXYC$ to the area of $\triangle AXB$ is $111 : 1$, compute $\frac{BC}{BA}$.

Answer: 10

Solution: Let $\frac{BC}{BA} = n$. Now, note that $BY = nAX$, and $\angle XBY = 180^\circ - \angle AXB - \angle CBY = 180^\circ - \angle AXB - \angle BAX = \angle AXB$. Then, by the Law of Sines, we have

$$\begin{aligned} [XBY] &= \frac{1}{2}(BX)(BY)\sin\angle XBY \\ &= n(AX)(BX)\sin\angle AXB \\ &= n[AXB]. \end{aligned}$$

Also, $[BYC] = n^2[AXB]$. It follows

$$\frac{[AXYC]}{[AXB]} = n^2 + n + 1$$

and $n^2 + n + 1 = 111$. By the quadratic formula, $n = \boxed{10}$.

16. Aidan has five final exams to take during finals week, each on a different weekday. During finals week, there are heavy storms and there is a 48.8% chance of a tree on campus falling down at some point in any given 24-hour period, where the probability of a tree falling down is uniform for the entire week and independent at different instances in time (i.e., a tree falling down at 9 AM does not affect the probability a tree falls down at 9:05 AM). On each day, if a tree falls down at any point between 9 AM and 5 PM, then Aidan's final for that day is canceled. What is the probability that at least two of his finals are canceled?

Answer: $\mathbf{\frac{821}{3125}}$

Solution: Let p be the probability that a tree falls down at some point in any given 8-hour period. Then the probability that no tree falls for any given 24-hour period is $(1-p)^3 = 1 - 0.488 = 0.512 = \frac{64}{125} = \left(\frac{4}{5}\right)^3$, so $p = \frac{1}{5}$. Then, on each weekday the probability that a tree falls down at some point between 9 AM and 5 PM is $\frac{1}{5}$. We use complementary counting and compute the probability that none or only one of Aidan's finals is canceled. The probability none are canceled is $\left(\frac{4}{5}\right)^5 = \frac{1024}{3125}$ and the probability that exactly one is canceled is $5 \cdot \frac{1}{5} \cdot \left(\frac{4}{5}\right)^4 = \frac{1280}{3125}$. Our answer then is $1 - \frac{1024}{3125} - \frac{1280}{3125} = \boxed{\frac{821}{3125}}$.

17. Suppose we have a triangle $\triangle ABC$ with $AB = 12$, $AC = 13$, and $BC = 15$. Let I be the incenter of triangle ABC. We draw a line through I parallel to BC intersecting AB at point D and AC at point E. What is the perimeter of triangle $\triangle ADE$?

Answer: 25

Solution: Notice that $\angle IBA = \angle IBC$ by property of incenter while $\angle IBC = \angle BID$ because DE is parallel to BC. Likewise, we also have $\angle ICA = \angle ICB = \angle CIE$. Then we get isosceles triangles $\triangle IDB$ and $\triangle IEC$. This gives $DE = ID + IE = DB + EC$. Then, the perimeter of triangle ADE is $AD + AE + DE = AD + DB + AE + EC = AB + AC = 12 + 13 = \boxed{25}$.

18. Ryan chooses five subsets S_1, S_2, S_3, S_4, S_5 of $\{1, 2, 3, 4, 5, 6, 7\}$ such that $|S_1| = 1, |S_2| = 2, |S_3| = 3, |S_4| = 4$, and $|S_5| = 5$. Moreover, for all $1 \le i < j \le 5$, either $S_i \cap S_j = S_i$ or $S_i \cap S_j = \emptyset$ (in other words, the intersection of S_i and S_j is either S_i or the empty set). In how many ways can Ryan select the sets?

Answer: 11760

Solution: Let $S = \{1, 2, 3, 4, 5, 6, 7\}$. Note that $S_i \cap S_j = S_i$ is equivalent to $S_i \subseteq S_j$. We use constructive counting and select S_5 first, and then move down to S_1. First, note that there are $\binom{7}{5}$ ways to choose S_5. Now, S_4 must be a subset of S_5 (as if $S_4 \cap S_5 = \emptyset$ then $|S| \ge 4 + 5 = 9$), so there are $\binom{5}{4}$ ways to choose S_4. Moreover, if S_3 is not a subset of S_4, then $S_3 = S - S_4$. But then $S_3 \cap S_5 \ne S_3$ and $S_3 \cap S_5 \ne \emptyset$, a contradiction. Thus $S_3 \subseteq S_4 \subseteq S_5$ and it follows that there are $\binom{4}{3}$ ways to choose S_3. Now, either S_2 is a subset of S_3, or $S_2 \cap S_3 = \emptyset$. In the first case, there are $\binom{3}{2}$ ways to choose S_2. In the latter case, since S_2 is not a subset of S_3, it cannot be a subset of S_4 and S_5. It follows that $S_2 \cap S_4, S_2 \cap S_5 = \emptyset$ and $S_2 = S - S_5$. Thus, there are $\binom{3}{2} + 1$ total ways to select S_2. Finally, note that S_1 can be any of $\{1, 2, 3, 4, 5, 6, 7\}$. The answer is

$$\binom{7}{5}\binom{5}{4}\binom{4}{3}\left(\binom{3}{2} + 1\right)7$$

$$= 21 \cdot 5 \cdot 4 \cdot 4 \cdot 7 = \boxed{11760}.$$

19. Bernie has an infinite supply of Nerds and Smarties with the property that eating one Nerd increases his IQ by 10 and eating one Smartie increases his IQ by 14. If Bernie currently has an IQ of 99, how many IQ values between 100 and 200, inclusive, can he achieve by eating Nerds and Smarties?

Answer: 38

Solution: Note that Bernie can only achieve odd IQ values since $\gcd(10, 14) = 2$. There are 50 odd numbers between 100 and 200. Then, 5 and 7 are relatively prime, so we can apply the Chicken McNugget Theorem to see that the largest even IQ increase that cannot be achieved with Nerds and Smarties is $2(5 \cdot 7 - 5 - 7) = 46 < 200 - 99$. Also by the theorem, the number of values that cannot be achieved is $\frac{(5-1)(7-1)}{2} = 12$. Thus, the number of different values that can be achieved is $50 - 12 = \boxed{38}$.

20. If the sum of the real roots x to each of the equations

$$2^{2x} - 2^{x+1} + 1 - \frac{1}{k^2} = 0$$

for $k = 2, 3, \ldots, 2023$ is N, what is 2^N?

Answer: $\mathbf{\frac{1012}{2023}}$

Solution: Define $y = 2^x$. Then, we can define the quadratic as $y^2 - 2y + 1 - \frac{1}{k^2}$. Through quadratic formula or inspection, we notice that this quadratic can be factored as $(y-(1-\frac{1}{k}))(y-(1+\frac{1}{k}))$. Hence, $y = 1 \pm \frac{1}{k}$. Thus, $2^x = 1 \pm \frac{1}{k} \to x = \log_2(1 \pm \frac{1}{k})$.

Note that the sum of the two solutions to a single equation is $\log_2\left(1 - \frac{1}{k^2}\right) = \log_2\left(\frac{k^2-1}{k^2}\right) = \log_2\left(\frac{(k-1)(k+1)}{k^2}\right)$. The sum of all solutions to the equations is then

$$\begin{aligned}
N &= \log_2\left(\frac{1\cdot 3}{2^2}\right) + \log_2\left(\frac{2\cdot 4}{3^2}\right) + \ldots + \log_2\left(\frac{2022\cdot 2024}{2023^2}\right) \\
&= \log_2\left(\frac{1\cdot 3}{2^2}\cdot\frac{2\cdot 4}{3^2}\cdot\ldots\cdot\frac{2022\cdot 2024}{2023^2}\right) \\
&= \log_2\left(\frac{1\cdot 2024}{2\cdot 2023}\right) \\
&= \log_2\left(\frac{1012}{2023}\right).
\end{aligned}$$

We have $2^N = \boxed{\frac{1012}{2023}}$.

21. Equilateral triangle $\triangle ABC$ is inscribed in circle Ω, which has a radius of 1. Let the midpoint of BC be M. Line AM intersects Ω again at point D. Let ω be the circle with diameter MD. Compute the radius of the circle that is tangent to BC on the same side of BC as ω, internally tangent to Ω, and externally tangent to ω.

Answer: $\mathbf{\frac{3}{16}}$

Solution: Denote the circle whose radius we want to compute as ω'. Let the center of Ω be O_1, the center of ω be O_2, and the center of ω' be O_3. Since O_1BM is a 30-60-90 triangle, we have $O_1M = \frac{1}{2}$. Then we see that the diameter of ω is $1 - \frac{1}{2} = \frac{1}{2}$, so the radius of ω is $\frac{1}{4}$. Let the radius of ω' be r. Let the foot of the perpendicular from O_3 to MD be point E. We have by the Pythagorean theorem

$$\begin{aligned}
O_3E &= \sqrt{O_3O_2^2 - O_2E^2} \\
&= \sqrt{\left(\frac{1}{4}+r\right)^2 - \left(\frac{1}{4}-r\right)^2} \\
&= \sqrt{r}.
\end{aligned}$$

Then, by the Pythagorean theorem in $\triangle O_1O_3E$ we have

$$O_3E^2 + O_1E^2 = O_1O_3^2,$$

which becomes

$$(\sqrt{r})^2 + \left(\frac{1}{2}+r\right)^2 = (1-r)^2.$$

Solving for r gives us $r = \boxed{\frac{3}{16}}$.

22. If (a, b) is a point on the circle centered at $(5, 0)$ with radius 4 in the xy-plane, compute the maximum possible value of $\frac{a^2+7b^2}{2a^2+b^2}$.

Answer: $\mathbf{\frac{121}{34}}$

Solution: This comes down to computing the maximum of $\frac{1+7s}{2+s} = 7 - \frac{13}{2+s}$, where s is a non-negative number denoting $(\frac{b}{a})^2$. In order to maximize this expression, $\frac{13}{2+s}$ should be minimized, i.e. $\left|\frac{b}{a}\right|$ be chosen as large as possible.

The value of $\frac{b}{a}$ is equal to the slope of the line passing through the origin and (a, b). The line with the greatest slope that passes through the origin and meets the circle must be tangent to the circle. Drawing tangent lines through the origin to the circle, one sees that the point (a, b) we desire lies at the vertex of the right angle of a right triangle with hypotenuse 5 and a leg of length 4. We find that $(a, b) = \left(\frac{9}{5}, \frac{12}{5}\right)$, so $\frac{b}{a} = \frac{4}{3}$. Plugging this into the expression, we get $7 - \frac{13}{2+\frac{16}{9}} = \boxed{\frac{121}{34}}$.

23. An ant begins walking while facing due east and every second turns 60° clockwise or counterclockwise, each with probability $\frac{1}{2}$. After the first turn the ant makes, what is the expected number of turns (not including the first turn) it makes before facing due east again?

Answer: 5

Solution: Note that the ant can only face 6 different directions at any point in time. Label them $0, 1, \ldots, 5$, with 0 being due east and the rest following in counterclockwise order. Without loss of generality, assume that the first turn the ant makes is in the counterclockwise direction. Let $E(i)$ be the expected number of turns to return to due east when facing direction i. We know that $E(0) = 0$, and we solve for $E(1)$ using the relations between $E(i)$ and $E(i-1), E(i+1)$:

$$\begin{aligned}
E(1) &= \frac{E(0)}{2} + \frac{E(2)}{2} + 1 \Rightarrow E(2) = 2E(1) - 2 \\
E(2) &= \frac{E(1)}{2} + \frac{E(3)}{2} + 1 \Rightarrow 2E(1) - 2 = \frac{E(1)}{2} + \frac{E(3)}{2} + 1 \\
&\qquad\qquad\qquad\qquad \Rightarrow E(3) = 3E(1) - 6 \\
E(3) &= \frac{E(2)}{2} + \frac{E(4)}{2} + 1 \Rightarrow 3E(1) - 6 = \frac{E(2)}{2} + \frac{E(4)}{2} + 1 \\
&\qquad\qquad\qquad\qquad \Rightarrow E(4) = 4E(1) - 12.
\end{aligned}$$

Now, note that $E(5) = E(1)$ by symmetry. We have

$$E(5) = \frac{E(4)}{2} + \frac{E(0)}{2} + 1 \Rightarrow E(1) = 2E(1) - 6 + 1,$$

which gives us $E(1) = \boxed{5}$.

24. Equilateral triangle $\triangle ABC$ has side length 12 and equilateral triangles of side lengths $a, b, c < 6$ are each cut from a vertex of $\triangle ABC$, leaving behind an equiangular hexagon $A_1A_2B_1B_2C_1C_2$, where A_1 lies on AC, A_2 lies on AB, and the rest of the vertices are similarly defined. Let A_3 be the midpoint of A_1A_2 and define B_3, C_3 similarly. Let the center of $\triangle ABC$ be O. Note that OA_3, OB_3, OC_3 split the hexagon into three pentagons. If the sum of the areas of the equilateral triangles cut out is $18\sqrt{3}$ and the ratio of the areas of the pentagons is $5 : 6 : 7$, what is the value of abc?

Answer: $64\sqrt{3}$

Solution: For the pentagon determined by OA_3 and OB_3, we can split it into $\triangle OA_3A_2, \triangle OA_2B_1, \triangle OB_1B_3$ and determine its area as

$$\frac{1}{2}\cdot\frac{a}{2}\cdot\left(4\sqrt{3}-\frac{a\sqrt{3}}{2}\right)+\frac{1}{2}\cdot 2\sqrt{3}\cdot(12-a-b)+\frac{1}{2}\cdot\frac{b}{2}\cdot\left(4\sqrt{3}-\frac{b\sqrt{3}}{2}\right)=\frac{\sqrt{3}}{8}(96-a^2-b^2).$$

Similarly, the areas of the other pentagons are $\frac{\sqrt{3}}{8}(96-b^2-c^2)$ and $\frac{\sqrt{3}}{8}(96-c^2-a^2)$. From the fact that the sum of the areas of the equilateral triangles cut out is $18\sqrt{3}$, we find that $a^2+b^2+c^2=\frac{4}{\sqrt{3}}\cdot 18\sqrt{3}=72$. Then,

$$96-a^2-b^2+96-b^2-c^2+96-c^2-a^2=288-2(a^2+b^2+c^2)=288-144=144.$$

Dividing 144 into the ratio $5:6:7$ gives $40, 48, 56$ as values for $96-a^2-b^2, 96-b^2-c^2, 96-c^2-a^2$. Then, we have $56, 48, 40$ for the values of $a^2+b^2, b^2+c^2, c^2+a^2$, which we can use along with $a^2+b^2+c^2=72$ to find that a, b, c are $4, 2\sqrt{6}, 4\sqrt{2}$ (in any order). Then, $abc=4\cdot 2\sqrt{6}\cdot 4\sqrt{2}=\boxed{64\sqrt{3}}$.

25. Consider a sequence $F_0=2$, $F_1=3$ that has the property $F_{n+1}F_{n-1}-F_n^2=(-1)^n\cdot 2$. If each term of the sequence can be written in the form $a\cdot r_1^n+b\cdot r_2^n$, what is the positive difference between r_1 and r_2?

Answer: $\frac{\sqrt{17}}{2}$

Solution: Listing out the first few terms of the sequence, we have $F_0=2, F_1=3, F_2=\frac{7}{2}, F_3=\frac{19}{4}, F_4=\frac{47}{8}$. Note that the terms of the sequence satisfy the recursive relation $F_{n+1}=\frac{F_n}{2}+F_{n-1}$. We will prove this inductively. Suppose that we already know that the property given in the problem and the recursive relation are satisfied for all F_n with $n\le k$. Then, we want to show that if $F_{k+1}=\frac{F_k}{2}+F_{k-1}$ then $F_{k+1}F_{k-1}-F_k^2=(-1)^k\cdot 2$. We have $F_{k+1}F_{k-1}-F_k^2=\frac{F_kF_{k-1}}{2}+F_{k-1}^2-F_k^2$. Note that $F_kF_{k-2}-F_{k-1}^2=(-1)^{n-1}\cdot 2 \Rightarrow F_{k-1}^2=F_kF_{k-2}+(-1)^n\cdot 2$. So,

$$\begin{aligned}\frac{F_kF_{k-1}}{2}+F_{k-1}^2-F_k^2&=\frac{F_kF_{k-1}}{2}+F_kF_{k-2}+(-1)^n\cdot 2-F_k^2\\&=F_k(\frac{F_{k-1}}{2}+F_{k-2}-F_k)+(-1)^n\cdot 2\\&=F_k\cdot 0+(-1)^n\cdot 2,\end{aligned}$$

which proves our claim. Now we know that the characteristic equation of the recurrence is $x^2=\frac{x}{2}+1$, and solving for x we get $x=\frac{1\pm\sqrt{17}}{4}$. These are the values of r_1 and r_2, so their positive difference is $\boxed{\frac{\sqrt{17}}{2}}$.