1. For all positive integers $n > 1$, let $f(n)$ denote the largest odd proper divisor of n (a proper divisor of n is a positive divisor of n except for n itself). Given that $N = 20^{23} \cdot 23^{20}$, compute

$$\frac{f(N)}{f(f(f(N)))}.$$

Answer: 25

Solution: Let $n > 1$ be a positive integer. If n is even, note that $f(n) = \frac{n}{2^{v(n)}}$, where $v(n)$ is the largest integer k such that 2^k divides n. Otherwise, if $n > 1$ is odd, we have $f(n) = \frac{n}{p(n)}$, where $p(n)$ is the smallest odd prime factor of n (which exists since $n > 1$ and n is odd). Using these observations, we find that $f(N) = 5^{23} \cdot 23^{20}$, $f(f(N)) = 5^{22} \cdot 23^{20}$, and $f(f(f(N))) = 5^{21} \cdot 23^{20}$. Our answer is

$$\frac{5^{23} \cdot 23^{20}}{5^{21} \cdot 23^{20}} = \boxed{25}.$$

2. A 3×3 grid is to be painted with three colors (red, green, and blue) such that

(i) no two squares that share an edge are the same color and

(ii) no two corner squares on the same edge of the grid have the same color.

As an example, the upper-left and bottom-left squares cannot both be red, as that would violate condition (ii). In how many ways can this be done? (Rotations and reflections are considered distinct colorings.)

Answer: 24

Solution: Let A be the upper-left corner and let the other corners be B, C, and D in counter-clockwise order. Now, let a denote the color on A. Then if the corners adjacent to A have the same color, say b, the last corner has color a or $c \neq a, b$. In the first case, the squares between A and B, B and C, C and D, D and A are determined to be of color c, while the center square still has 2 options (a or b). In the second case, corner C is of color c, while every other square gets uniquely determined. Now, in the last case, the corners adjacent to A are of different colors b, c (WLOG b at corner B and c at corner D), implying the last corner has color a and all the remaining squares are determined. Thus, for a fixed triplet (a, b, c), there are a total of $2 + 1 + 1 = 4$ different configurations of colors. As there are $3! = 6$ ways to determine (a, b, c), the answer is $6 \cdot 4 = \boxed{24}$.

3. How many trailing zeros does the value

$$300 \cdot 305 \cdot 310 \cdots 1090 \cdot 1095 \cdot 1100$$

end with?

Answer: 161

Solution: Rewrite the expression as

$$300 \cdot 305 \cdot 310 \cdots 1090 \cdot 1095 \cdot 1100 = 5^{161} \cdot \frac{220!}{59!}.$$

By Legendre's formula, there are

$$\left\lfloor \frac{59}{5} \right\rfloor + \left\lfloor \frac{59}{25} \right\rfloor = 13$$

factors of 5 in 59! and

$$\left\lfloor \frac{59}{2} \right\rfloor + \left\lfloor \frac{59}{4} \right\rfloor + \left\lfloor \frac{59}{8} \right\rfloor + \left\lfloor \frac{59}{16} \right\rfloor + \left\lfloor \frac{59}{32} \right\rfloor = 54$$

factors of 2 in 59!. Morever, there are

$$\left\lfloor \frac{220}{5} \right\rfloor + \left\lfloor \frac{220}{25} \right\rfloor + \left\lfloor \frac{220}{125} \right\rfloor = 53$$

factors of 5 in 220! and

$$\left\lfloor \frac{220}{2} \right\rfloor + \left\lfloor \frac{220}{4} \right\rfloor + \left\lfloor \frac{220}{8} \right\rfloor + \left\lfloor \frac{220}{16} \right\rfloor + \left\lfloor \frac{220}{32} \right\rfloor + \left\lfloor \frac{220}{64} \right\rfloor + \left\lfloor \frac{220}{128} \right\rfloor = 215$$

factors of 2 in 220!. Thus there are

$$161 + 53 - 13 = 201$$

factors of 5 in the product and

$$215 - 54 = 161$$

factors of 2 in the product. Since $161 < 201$, it follows the answer is $\boxed{161}$.

4. Michelle is drawing segments in the plane. She begins from the origin facing up the y-axis and draws a segment of length 1. Now, she rotates her direction by 120°, with equal probability clockwise or counterclockwise, and draws another segment of length 1 beginning from the end of the previous segment. She then continues this until she hits an already drawn segment. What is the expected number of segments she has drawn when this happens?

Answer: 4

Solution: Michelle's drawing process is that of a random walk on the triangulated lattice. Notice that if at any point Michelle has two of the same rotation in a row, she will necessarily hit an already drawn segment by constructing an equilateral triangle. Note as well that if she alternates rotations every step, then she will go to infinity and never intersect a prior segment.

The problem thus reduces to finding the expected time of first occurrence of two adjacent heads or tails in a series of coin tosses. Let $f(X)$ denote this expected time if the last toss was X. Then, $f(H) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (1 + f(T)) = 1 + \frac{1}{2}f(T)$ and $f(T) = 1 + \frac{1}{2}f(H)$. This implies that $f(T) = f(H) = 2$. Hence, our final answer is $2 + 2 = \boxed{4}$ (where we add the first segment and the first rotated segment).

5. Ryan chooses five subsets S_1, S_2, S_3, S_4, S_5 of $\{1, 2, 3, 4, 5, 6, 7\}$ such that $|S_1| = 1, |S_2| = 2, |S_3| = 3, |S_4| = 4$, and $|S_5| = 5$. Moreover, for all $1 \leq i < j \leq 5$, either $S_i \cap S_j = S_i$ or $S_i \cap S_j = \emptyset$ (in other words, the intersection of S_i and S_j is either S_i or the empty set). In how many ways can Ryan select the sets?

Answer: 11760

Solution: Let $S = \{1, 2, 3, 4, 5, 6, 7\}$. Note that $S_i \cap S_j = S_i$ is equivalent to $S_i \subseteq S_j$. We use constructive counting and select S_5 first, and then move down to S_1. First, note that there are $\binom{7}{5}$ ways to choose S_5. Now, S_4 must be a subset of S_5 (as if $S_4 \cap S_5 = \emptyset$ then $|S| \geq 4 + 5 = 9$), so there are $\binom{5}{4}$ ways to choose S_4. Moreover, if S_3 is not a subset of S_4, then $S_3 = S - S_4$. But then $S_3 \cap S_5 \neq S_3$ and $S_3 \cap S_5 \neq \emptyset$, a contradiction. Thus $S_3 \subseteq S_4 \subseteq S_5$ and it follows that there are $\binom{4}{3}$ ways to choose S_3. Now, either S_2 is a subset of S_3, or $S_2 \cap S_3 = \emptyset$. In the first case, there are $\binom{3}{2}$ ways to choose S_2. In the latter case, since S_2 is not a subset of S_3, it cannot be a subset of S_4 and S_5. It follows that $S_2 \cap S_4, S_2 \cap S_5 = \emptyset$ and $S_2 = S - S_5$. Thus, there are $\binom{3}{2} + 1$ total ways to select S_2. Finally, note that S_1 can be any of $\{1, 2, 3, 4, 5, 6, 7\}$. The answer is

$$\binom{7}{5}\binom{5}{4}\binom{4}{3}\left(\binom{3}{2}+1\right)7$$
$$= 21 \cdot 5 \cdot 4 \cdot 4 \cdot 7 = \boxed{11760}.$$

6. We say that an integer $x \in \{1, \cdots, 102\}$ is *square-ish* if there exists some integer n such that $x \equiv n^2 + n \pmod{103}$. Compute the product of all square-ish integers modulo 103.

Answer: 52

Solution: Note that $n^2 + n \equiv (n + 52)^2 - 52^2$. The set of square-ish residues is equal to the set of residues of the form $x^2 - 52^2$. It is tempting to simply multiply out $\prod_x x^2 - 52^2$, but we must prevent multiplicity. Aside from the case $x = 0$, each residue has two values of x corresponding to it, x and $103 - x$, so to prevent multiplicity it suffices to calculate

$$\begin{aligned}\prod_{x=0}^{50}(x^2 - 52^2) &\equiv \prod_{x=0}^{50}(x - 52)(x + 52)\\ &\equiv (-52)(-51)\cdots(-2)\cdot(52)(53)\cdots(102)\\ &\equiv (-2)(-3)\cdots(-52)\cdot(52)(53)\cdots(102)\\ &\equiv (-1)^{51}102!\cdot 52 \pmod{103}.\end{aligned}$$

By Wilson's Theorem, $102! \equiv -1$. Thus, the residue comes out to be $(-1)^{52} \cdot 52 = \boxed{52}$.

7. Let S be the number of bijective functions $f : \{0, 1, \dots, 288\} \to \{0, 1, \dots, 288\}$ such that $f((m+n) \bmod 17)$ is divisible by 17 if and only if $f(m) + f(n)$ is divisible by 17. Compute the largest positive integer n such that 2^n divides S.

Answer: 270

Solution: Since f is bijective, there exists some $m \in \{0, 1, \dots, 288\}$ such that $f(m) \equiv 0 \bmod 17$. For any integer k, since $f(m + 17k \bmod 17) = f(m + 0) \equiv 0 \bmod 17$, by the condition given in the problem we also know that $f(m) + f(17k) \equiv 0 \bmod 17$. This gives us $f(17k) \equiv 0 \bmod 17$ for any integer k.

So, $m + n \equiv 0 \bmod 17$ if and only if $f(m) + f(n) \equiv 0 \bmod 17$. There are 8 pairs of residues modulo 17 that sum to 17 (for example, $(1, 16)$ is such a pair). Each pair is mapped by f to another pair, so there are $8!$ ways to order them, and the residues within each pair can be switched, giving us a factor of 2^8.

Then, in $\{0, 1, \dots, 288\}$ there are 17 numbers for each residue modulo 17. Once the mappings of residues modulo 17 have been determined, there are $17!$ ways to map each of the numbers with the same residue. Thus, $S = 2^8 \cdot 8! \cdot (17!)^{17}$. The exponent of the largest power of 2 that divides S is $8 + (4 + 2 + 1) + 17(8 + 4 + 2 + 1) = \boxed{270}$.

8. Define the Fibonacci numbers via $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$.

Olivia flips two fair coins at the same time, repeatedly, until she has flipped a tails on both, not necessarily on the same throw. She records the number of pairs of flips c until this happens (not including the last pair, so if on the first flip both coins turned up tails c would be 0). What is the expected value of F_c?

Answer: $\mathbf{\frac{19}{11}}$

Solution: Let a, b be the number of flips of each coin until tails. Then, we are looking for the expected value of $\max(F_a, F_b)$.

Suppose that $a = 0$. Then, this expected value is

$$\frac{F_0}{2} + \frac{F_1}{2^2} + \frac{F_2}{2^3} + \cdots,$$

that is, the regular expected value. However, if $a = n$ then the expected value becomes

$$\frac{F_n}{2} + \frac{F_n}{2^2} + \cdots + \frac{F_n}{2^{n+1}} + \frac{F_{n+1}}{2^{n+2}} + \cdots = F_n \cdot \left(1 - \frac{1}{2^{n+1}}\right) + \frac{F_{n+1}}{2^{n+2}} + \cdots$$

that is, the first n cases have numerator transformed into F_n. So, let f_n be the expected value given that $a = n$. Then, the quantity we are looking for is

$$E = \frac{f_0}{2} + \frac{f_1}{2^2} + \frac{f_2}{2^3} + \cdots$$

Let's look at the contribution of F_n to this sum, since F_n can only appear in $f_0, f_1, \dots, f_n$. In the first n of these, F_n has a coefficient of $\frac{1}{2^{n+1}}$ and in the last one it has $1 - \frac{1}{2^{n+1}}$. So, its contribution is

$$\frac{1}{2^{n+1}} \cdot \left(1 - \frac{1}{2^{n+1}}\right) + \frac{1}{2^{n+1}} \sum_{i=0}^{n-1} \frac{1}{2^{i+1}} = \frac{1}{2^{n+1}} \cdot \left(1 - \frac{1}{2^{n+1}} + 1 - \frac{1}{2^n}\right) = \frac{2}{2^{n+1}} - \frac{3}{4^{n+1}}.$$

Therefore, the sum we are looking for is

$$E = \sum_{n=0}^{\infty} \left(\frac{2}{2^{n+1}} - \frac{3}{4^{n+1}}\right) F_n.$$

It is well known (or done via generating functions) that

$$\sum_{n=0}^{\infty} F_n x^{n+1} = \frac{x^2}{1 - x - x^2}$$

so our answer is

$$E = 2 \cdot \frac{\left(\frac{1}{2}\right)^2}{1 - \frac{1}{2} - \left(\frac{1}{2}\right)^2} - 3 \cdot \frac{\left(\frac{1}{4}\right)^2}{1 - \frac{1}{4} - \left(\frac{1}{4}\right)^2} = \boxed{\frac{19}{11}}.$$

9. Suppose a and b are positive integers with a curious property: $(a^3 - 3ab + \frac{1}{2})^n + (b^3 + \frac{1}{2})^n$ is an integer for at least 3, but at most finitely many different choices of positive integers n. What is the least possible value of $a + b$?

Answer: 6

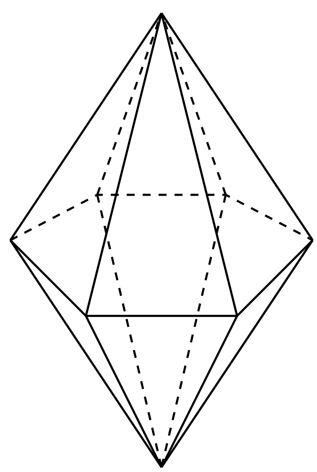
Solution: Observe that the condition $(a^3 - 3ab + \frac{1}{2})^n + (b^3 + \frac{1}{2})^n$ being an integer is equivalent to $(2a^3 - 6ab + 1)^n + (2b^3 + 1)^n$ being divisible by 2^n. If n is even, both powers are equal to 1 modulo 4, so the expression is never divisible by 4, contradiction.

Meanwhile if n is odd, we can factor out $(2a^3-6ab+1)+(2b^3+1)$ from the expression. The other factor is a sum of n odd numbers, thus is odd. We thus demand that $(2a^3-6ab+1)+(2b^3+1) = 2(a^3+b^3+1-3ab)$ be divisible by at least 3, but at most finitely many odd powers of 2.

If $2^n \mid 2(a^3+b^3+1-3ab)$, of course all powers of 2 less than n also divide $2(a^3+b^3+1-3ab)$. So it suffices to make $(a^3+b^3+1-3ab)$ divisible by 16 (which would mean that $2, 8, 32$ divide $2(a^3+b^3+1-3ab)$), but nonzero (if the expression is equal to zero, which is the case when $(a,b) = (1,1)$, an infinite number of powers of 2 will divide it). Factoring, $(a^3+b^3+1-3ab) = (a+b+1)(a^2+b^2+1-ab-a-b)$.

Because a^2-a, b^2-b are always even, $(a^2+b^2+1-ab-a-b)$ is even iff $ab-1$ is even iff a, b are both odd, in which case $a+b+1$ is odd. So, either $16 \mid a+b+1$, or $16 \mid (a^2+b^2+1-ab-a-b)$. In the former case, we have $a+b$ at least 15. In the latter case, setting $a=1$ and experimenting, we see that $(a,b) = (1,5)$ is a valid pair, whereas any pair with smaller sum will not work. Thus $1+5 = \boxed{6}$ is the solution desired.

10. Colin has a peculiar 12-sided dice: it is made up of two regular hexagonal pyramids. Colin wants to paint each face one of three colors so that no two adjacent faces *on the same pyramid* have the same color. How many ways can he do this? Two paintings are considered identical if there is a way to rotate or flip the dice to go from one to the other. Faces are adjacent if they share an edge.



Answer: 405

Solution: Consider how we can rotate the dice. We can rotate either about the axis containing the vertices with six faces adjoining them, or around one of the vertices with four faces. Note that doing two rotations about the "central" vertices is equivalent to doing none, and rotations are just reversed when in a rotated state. Furthermore, the flip operation is a central rotation combined with three top rotations. Hence, there are 12 symmetries. Let G be the set of these symmetries and for each $g \in G$, let $f(g)$ the number of ways to draw on the faces such that applying g does nothing (for example, if g is a rotation by one face, then $f(g)$ is 0 since it would imply two adjacent faces are equal). By Burnside's Lemma, the answer then is $\frac{1}{12}\sum_{g\in G} f(g)$.

For convenience, label the top pyramid as having faces $a_1, a_2, \ldots, a_6$ and the bottom pyramid as having $b_1, b_2, \ldots, b_6$. Let $c(x)$ be the color of the face x.

With this in mind, let's compute $f(g)$ for each g. First, we look at the symmetries with no flips (that is, just rotations). As stated previously, if the rotation is by 1 or 5 then $f(g)$ is 0 since two adjacent faces will be equal. Next, consider a rotation by 2 or 4. These imply that every other face in each of the pyramids is equal. Therefore, there are $(3 \cdot 2)^2 = 36$ possible colorings.

Next, let's look at a rotation by 3. Then, $c(a_1), c(a_2), c(a_3)$ must all be distinct. There are again $(3!)^2 = 36$ possible colorings.

Finally, among the rotations the final case is no rotation (that is, the identity symmetry). In this case, each of the pyramids must have no two adjacent faces having the same color (with no other restrictions). Let $f(n)$ be the number of ways to do this if the pyramid had n faces instead of just 6. If $c(a_1) \neq c(a_{n-1})$, then we $c(a_n)$ has only one possibility. If $c(a_1) = c(a_{n-1})$, then $c(a_1) \neq c(a_{n-2})$ and there are two possible assignments for $c(a_n), c(a_{n-1})$. Hence, $f(n) = f(n-1) + 2f(n-2)$. With the base cases $f(2) = 6, f(3) = 6$, we can quickly compute $f(4) = 18, f(5) = 30, f(6) = 66$ (alternatively, one can compute $f(n) = 2^n + 2 \cdot (-1)^n$). Hence, this case gives $66^2 = 4356$.

Now, consider each of the flip symmetries. In fact, each of these contribute 66: for example, just a flip implies that $a_1 = b_4, a_2 = b_3, a_3 = b_2, a_4 = b_1, a_5 = b_6, a_6 = b_5$. Hence, the two colorings must be identical up to reflection and translation.

Therefore, our answer is

$$\frac{1}{12}\left(36 \cdot 2 + 36 + 4356 + 66 \cdot 6\right) = \boxed{405}.$$