1. Compute
$$\frac{\int_{-\infty}^{\infty} x^{102}e^{1-x^4}\,\mathrm{d}x}{\int_{-\infty}^{\infty} x^{98}e^{1-x^4}\,\mathrm{d}x}.$$

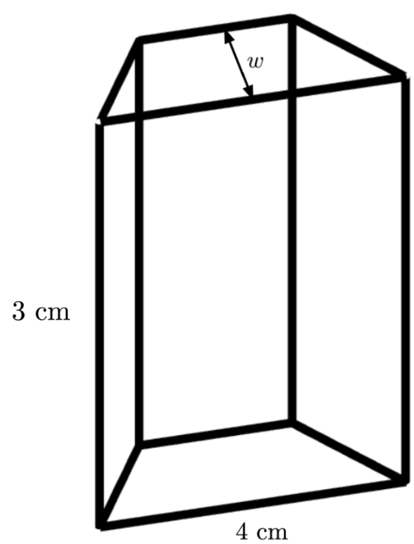
Answer: $\frac{99}{4}$

Solution: Notice that the 1 in the exponent doesn't matter, so we may cancel it out. Now, for any n, $\int_{-\infty}^{\infty} x^{n+3}e^{-x^4}\,\mathrm{d}x$, using integration by parts with $u = x^n$ and $\mathrm{d}v = x^3e^{-x^4}$, is

$$\int_{-\infty}^{\infty} x^{n+3}e^{-x^4}\,\mathrm{d}x = -\frac{1}{4}x^n e^{-x^4}\bigg|_{x=-\infty}^{\infty} + \frac{n}{4}\int_{-\infty}^{\infty} x^{n-1}e^{-x^4}\,\mathrm{d}x = \frac{n}{4}\int_{-\infty}^{\infty} x^{n-1}e^{-x^4}\,\mathrm{d}x.$$

Taking $n = 99$ thus gives the answer as $\boxed{\frac{99}{4}}$.

2. A block of cheese in the shape of a triangular prism with height 3 cm and equilateral bases of side length 4 cm is grated along a plane parallel to one of its lateral (non-base) faces. The grater is initially placed at the edge of the prism opposite the lateral face, and over time the prism is truncated along this edge. Call the "width" of the cheese the distance between the grater and the lateral face. The cheese that is grated falls into a funnel that is a square pyramid with height $3\sqrt{3}$ cm and base of side length 4 cm; the cheese falls through the square base at the top toward the vertex. Assume that the grated cheese is fine enough to fill the funnel continuously. If the width of the cheese decreases at a rate of 1 cm/s when the width is $2\sqrt{3} - \sqrt{2}$ cm, the height of the cheese in the funnel increases at a rate of d cm/s. Compute d.



Answer: $\frac{\sqrt{6}}{2}$

Solution: Let w_0 be the initial width of the cheese and w the current width. The volume of cheese that has not been grated is $V_1 = 12\sqrt{3}\left(1 - \left(\frac{w_0-w}{w_0}\right)^2\right)$, since the initial volume of the cheese is $12\sqrt{3}$. Taking the derivative with respect to time gives us $\frac{dV_1}{dt} = 24\sqrt{3} \cdot \frac{w_0-w}{w_0} \cdot \frac{1}{w_0} \cdot \frac{dw}{dt}$. We have that $w_0 = 2\sqrt{3}$, the height of an equilateral triangle with side length 4.

Let $s_f = 4$ and $h_f = 3\sqrt{3}$ be the base side length and height of the funnel, and let h be the height of the cheese in the funnel. The volume of cheese in the funnel is $V_2 = 16\sqrt{3}\left(\frac{h}{h_f}\right)^3$, since $16\sqrt{3}$ is the volume of the funnel. Taking the derivative with respect to time gives us $\frac{dV_2}{dt} = 48\sqrt{3}\left(\frac{h}{h_f}\right)^2 \cdot \frac{1}{h_f} \cdot \frac{dh}{dt}$. This gives us

$$\begin{aligned}\frac{dh}{dt} &= \frac{dV_2}{dt} \cdot \frac{h_f^3}{48\sqrt{3}h^2} \\ &= -\frac{dV_1}{dt} \cdot \frac{h_f^3}{48\sqrt{3}h^2} \\ &= -24\sqrt{3} \cdot \frac{w_0 - w}{w_0} \cdot \frac{1}{w_0} \cdot \frac{dw}{dt} \cdot \frac{h_f^3}{48\sqrt{3}h^2}.\end{aligned}$$

When $w = 2\sqrt{3} - \sqrt{2}$, the volume of cheese that has been grated is $\left(\frac{\sqrt{2}}{2\sqrt{3}}\right)^2 \cdot 12\sqrt{3} = 2\sqrt{3}$. To solve for h, we have $2\sqrt{3} = 16\sqrt{3}\left(\frac{h}{h_f}\right)^3$, so $h = \frac{h_f}{2} = \frac{3\sqrt{3}}{2}$. Then,

$$\begin{aligned}\frac{dh}{dt} &= -24\sqrt{3} \cdot \frac{\sqrt{2}}{2\sqrt{3}} \cdot \frac{1}{2\sqrt{3}} \cdot \frac{81\sqrt{3}}{48\sqrt{3} \cdot 27/4} \cdot \frac{dw}{dt} \\ &= -\frac{\sqrt{6}}{2} \cdot (-1) \\ &= \boxed{\frac{\sqrt{6}}{2}}.\end{aligned}$$

3. A frog hops on the real number line, starting from the origin. Each second, it moves right uniformly at random a distance between 0 and 1. There is an abyss between 1 and $\frac{5}{4}$, and if the frog lands there it will fall into the abyss. If the frog makes it to or passes 3 without falling in the abyss, then the frog is safe. What is the probability it is safe?

Answer: $\boldsymbol{e^{\frac{1}{4}} - \frac{e}{4}}$

Solution: Let $p(x)$ be the probability that the frog ends up safe, if starting from position x. Notice that the 3 is a red herring: in fact, if the frog lands anywhere after $\frac{5}{4}$ without landing in the abyss it will be safe.

If $\frac{1}{4} \le x \le 1$, then $p(x) = (x - \frac{1}{4}) + \int_x^1 p(y)\,\mathrm{d}y$: the probability of being safe is 0 for $1 \le x \le \frac{5}{4}$ and 1 for $x > \frac{5}{4}$.

Hence, we have $p'(x) = 1 - p(x)$ by the Second Fundamental Theorem of Calculus. Therefore, one can guess (there is a systematic way of doing this, but is not required) that $p(x) = 1 + Ce^{-x}$. Since $p(1) = \frac{3}{4}$, it follows that $C = -\frac{e}{4}$.

Now, when $0 \le x \le \frac{1}{4}$, then

$$p(x) = \int_x^1 p(y)\,\mathrm{d}y = \int_x^{\frac{1}{4}} p(y)\,\mathrm{d}y + \int_{\frac{1}{4}}^1 \left(1 - \frac{e}{4} \cdot e^{-y}\right)\,\mathrm{d}y.$$

Therefore, in this range, $p'(x) + p(x) = 0$ since the second term is a constant and so $p(x) = De^{-x}$. Since we are looking for $p(0)$, it suffices to compute D. As $De^{-\frac{1}{4}} = p(\frac{1}{4}) = 1 - \frac{e}{4} \cdot e^{-\frac{1}{4}} = 1 - \frac{e^{\frac{3}{4}}}{4}$, it follows that $D = \boxed{e^{\frac{1}{4}} - \frac{e}{4}}$ and we are done.