1. Compute the area of the polygon formed by connecting the roots of

$$x^{10} + x^9 + x^8 + x^6 + x^5 + x^4 + x^2 + x + 1$$

graphed in the complex plane with line segments in counterclockwise order.

Answer: $\mathbf{1 + \sqrt{3}}$

Solution: Note that the polynomial can be factored as $\frac{x^3-1}{x-1} \cdot \frac{x^{12}-1}{x^4-1}$. We can find the roots of polynomial by taking the roots of $(x^3-1)(x^{12}-1)$ and removing the roots of $(x-1)(x^4-1)$. This gives us $e^{i\pi/6}, e^{i\pi/3}, e^{i2\pi/3}, e^{i5\pi/6}, e^{i7\pi/6}, e^{i4\pi/3}, e^{i5\pi/3}, e^{i11\pi/6}$ as the roots. Note that the resulting polynomial consists of four isosceles triangles with sides of length 1 around an angle of 30° and four isosceles triangles with sides of length 1 around an angle of 60°. The total area of these triangles using the Law of Sines is $4 \cdot \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin 30° + 4 \cdot \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin 60° = \boxed{1 + \sqrt{3}}$.

2. $f(x)$ is a nonconstant polynomial. Given that $f(f(x)) + f(x) = f(x)^2$, compute $f(3)$.

Answer: 6

Solution: We use LHS to refer to the left hand side of the equation and RHS to refer to the right hand side. Let the degree of f be k. If $k \geq 2$, LHS has degree k^2 and RHS has degree $2k$, with equality only at $k = 2$. Thus, the degree is 2.

Let $f(x) = ax^2 + bx + c$. The LHS coefficient of x^4 is a^3, while the RHS coefficient of x^4 is a^2; equality gives $a = 1$. So $f(x) = x^2 + bx + c$. In addition, note that $f(x)$ divides $f(x)^2 - f(x) = f(f(x)) = f(x)^2 + bf(x) + c$, forcing $c = 0$.

Now, the condition given in the problem becomes $f(x)^2 + bf(x) + f(x) = f(x)^2 \to b = -1$. Hence, $f(x) = x^2 - x$, and $f(3) = \boxed{6}$.

3. Let $f(x) = x^3 - 6x^2 + \frac{25}{2}x - 7$. There is an interval $[a, b]$ such that for any real number x, the sequence $x, f(x), f(f(x)), \cdots$ is bounded (i.e., has a lower and upper bound) if and only if $x \in [a, b]$. Compute $(a - b)^2$.

Answer: 2

Solution: Note that $f(x) - x$ can be factored as $(x - 2)((x - 2)^2 - \frac{1}{2})$, so in particular, $2 \in [a, b]$ (since $f(2) - 2 = 0 \to 2 = f(2) = f(f(2)) = \cdots$).

For any x, write $x = 2+\delta$. Then, $f(x)-2 = (x-2)((x-2)^2-\frac{1}{2})+x-2 = \delta(\delta^2-\frac{1}{2})+\delta = \delta(\delta^2+\frac{1}{2})$. If $|\delta| > \frac{1}{\sqrt{2}}$, then $|f(x) - 2| > |\delta|c$ for some $c > 1$, and repeating this process (setting $f(x) = 2 + \delta'$ with $|\delta'| > |\delta|c$) shows divergence. Conversely, if $|\delta| < \frac{1}{\sqrt{2}}$, the factor c is smaller than 1. Since we are given that the converging set is closed, it follows that $[a, b] = 2 \pm \frac{1}{\sqrt{2}}$ and hence $a - b = \sqrt{2} \to (a - b)^2 = \boxed{2}$.