1. There exists a unique real value of x such that

$$(x + \sqrt{x})^2 = 16.$$

Compute x.

Answer: $\frac{9-\sqrt{17}}{2}$

Solution: In order for $\sqrt{x}$ to be defined, $x \geq 0$. Then $x + \sqrt{x} \geq 0$ and $x + \sqrt{x} = 4$. Letting $y = \sqrt{x}$, we get $y^2 + y - 4 = 0$ which by the quadratic formula has solutions $\frac{-1 \pm \sqrt{17}}{2}$. As $y = \sqrt{x} \geq 0$, it follows that $y = \frac{-1+\sqrt{17}}{2}$ and

$$x = y^2 = \frac{18 - 2\sqrt{17}}{4} = \boxed{\frac{9-\sqrt{17}}{2}}.$$

2. Compute the number of values of x in the interval $[-11\pi, -2\pi]$ that satisfy $\frac{5\cos(x)+4}{5\sin(x)+3} = 0$.

Answer: 4

Solution: The fraction is equal to zero when its numerator is equal to zero and its denominator is not equal to zero. The solutions to $5\cos(x) + 4 = 0$ are of the form $x = \pm \arccos(-4/5) + 2\pi k$ for integer k. The solutions to $5\sin(x) + 3 = 0$ are of the form $x = \pm \arcsin(-3/5) + 2\pi k$ for integer k. We see that every interval of the form $[2k\pi, (2k+1)\pi]$ has one solution to the given equation and intervals of the form $[(2k+1)\pi, (2k+2)\pi]$ have no solutions. Thus, there are $\boxed{4}$ solutions in the interval $[-11\pi, -2\pi]$.

3. Nathan has discovered a new way to construct chocolate bars, but it's expensive! He starts with a single 1×1 square of chocolate and then adds more rows and columns from there. If his current bar has dimensions $w \times h$ (w columns and h rows), then it costs w^2 dollars to add another row and h^2 dollars to add another column. What is the minimum cost to get his chocolate bar to size 20×20?

Answer: 5339

Solution: The optimal way to add rows and columns to the 1×1 chocolate to the 20×20 chocolate is to alternate adding rows and columns. (A rough proof of this is below.) If we do this, then the costs are 1^2 for the first row, plus 2^2 for the first column, plus 2^2 for the second row, plus $3^2 + 3^2 + 4^2 + \cdots$. The formula for the overall cost to get to $n \times n$ is $1^2 + 2 \cdot 2^2 + 2 \cdot 3^2 + \cdots + 2 \cdot (n-1)^2 + n^2$. The sum of the first n squares can be calculated as $\frac{n(n+1)(2n+1)}{6}$. Thus, we can simplify our desired sum to $\frac{n(n+1)(2n+1)}{3} - 1 - n^2$. For $n = 20$ this equals $\frac{20 \cdot 21 \cdot 41}{3} - 1 - 20^2 = \boxed{5339}$.

Proof: Assume $w > h$ (more columns than rows). Adding a column and then a row costs $h^2 + (w+1)^2$. Adding a row and then a column costs $w^2 + (h+1)^2$. Since $w > h$, we have $h^2 + (w+1)^2 = h^2 + w^2 + 2w + 1 > h^2 + w^2 + 2h + 1 = w^2 + (h+1)^2$. Therefore, it's always more optimal to add a row first in this case. We can see that alternating rows and columns is optimal.

4. If the sum of the real roots x to each of the equations

$$2^{2x} - 2^{x+1} + 1 - \frac{1}{k^2} = 0$$

for $k = 2, 3, \ldots, 2023$ is N, what is 2^N?

Answer: $\frac{\mathbf{1012}}{\mathbf{2023}}$

Solution: Define $y = 2^x$. Then, we can define the quadratic as $y^2 - 2y + 1 - \frac{1}{k^2}$. Through quadratic formula or inspection, we notice that this quadratic can be factored as $(y-(1-\frac{1}{k}))(y-(1+\frac{1}{k}))$. Hence, $y = 1 \pm \frac{1}{k}$. Thus, $2^x = 1 \pm \frac{1}{k} \to x = \log_2(1 \pm \frac{1}{k})$.

Note that the sum of the two solutions to a single equation is $\log_2\left(1 - \frac{1}{k^2}\right) = \log_2\left(\frac{k^2-1}{k^2}\right) = \log_2\left(\frac{(k-1)(k+1)}{k^2}\right)$. The sum of all solutions to the equations is then

$$\begin{aligned} N &= \log_2\left(\frac{1\cdot 3}{2^2}\right) + \log_2\left(\frac{2\cdot 4}{3^2}\right) + \ldots + \log_2\left(\frac{2022\cdot 2024}{2023^2}\right) \\ &= \log_2\left(\frac{1\cdot 3}{2^2}\cdot\frac{2\cdot 4}{3^2}\cdot\ldots\cdot\frac{2022\cdot 2024}{2023^2}\right) \\ &= \log_2\left(\frac{1\cdot 2024}{2\cdot 2023}\right) \\ &= \log_2\left(\frac{1012}{2023}\right). \end{aligned}$$

We have $2^N = \boxed{\frac{1012}{2023}}$.

5. Suppose $\alpha, \beta, \gamma \in \{-2, 3\}$ are chosen such that

$$M = \max_{x\in\mathbb{R}} \min_{y\in\mathbb{R}_{\geq 0}} \alpha x + \beta y + \gamma xy$$

is finite and positive (note: $\mathbb{R}_{\geq 0}$ is the set of nonnegative real numbers). What is the sum of the possible values of M?

Answer: $\frac{\mathbf{13}}{\mathbf{2}}$

Solution: We have

$$\max_{x\in\mathbb{R}} \min_{y\in\mathbb{R}_{\geq 0}} \alpha x + \beta y + \gamma xy = \max_{x\in\mathbb{R}} \min_{y\in\mathbb{R}_{\geq 0}} \alpha x + y(\beta + \gamma x)$$

Note that if $\beta + \gamma x < 0$, then by increasing y, the minimum could be arbitrarily small, so to maximize the value, it is never a good strategy to pick such an x. Thus, we will choose x such that $\beta + \gamma x \geq 0$, and this forces $y = 0$ as the best choice for y. This gives us

$$\max_{x\in\mathbb{R}} \min_{y\in\mathbb{R}_{\geq 0}} \alpha x + y(\beta + \gamma x) = \max_{x\in\mathbb{R}, \beta+\gamma x\geq 0} \alpha x.$$

The constraint $\beta + \gamma x \geq 0$ is equivalent to $\gamma x \geq -\beta$. Note that α and γ must not have the same sign, as otherwise by making x very large with the same sign as α and γ, we can satisfy the constraint and cause the value of αx to diverge.

In order for M to be positive, α and x must have the same sign. Then, γx is 0 or a negative value. From the constraint $\gamma x \geq -\beta$., we see that we must have $\beta \geq 0$, i.e. $\beta = 3$. The maximum possible value of x that satisfies the constraint is $-\frac{\beta}{\gamma}$, which gives us

$$\max_{x\in\mathbb{R}, \beta+\gamma x\geq 0} \alpha x = -\frac{\alpha\beta}{\gamma}.$$

The possible values of α/γ are $-2/3$ and $-3/2$. Therefore, the possible values of M are 2 or $9/2$, whose sum is $\boxed{13/2}$.

6. What is the area of the figure in the complex plane enclosed by the origin and the set of all points $\frac{1}{z}$ such that $(1-2i)z + (-2i-1)\overline{z} = 6i$?

Answer: $\frac{5\pi}{36}$

Solution 1: We can rewrite $(1-2i)z+(-2i-1)\overline{z} = 6i$ as $\frac{z-\overline{z}}{2i} = z+\overline{z}+3$. If we let $z = x+yi$, this is equivalent to the equation $y = 2x+3$. Suppose that a point $u = \frac{1}{z}$ where $(1-2i)z+(-2i-1)\overline{z} = 6i$. Let $u = v + wi$. Then, $z = \frac{1}{u} = \frac{v}{v^2+w^2} - \frac{w}{v^2+w^2}i$ and we must also have

$$-\frac{w}{v^2+w^2} = 2\frac{v}{v^2+w^2} + 3.$$

This can be rewritten as

$$\left(v+\frac{1}{3}\right)^2 + \left(w+\frac{1}{6}\right)^2 = \frac{5}{36}.$$

Note that we cannot allow $v^2 + w^2 = 0$ but the origin is still included in the set of points we consider given the problem statement. The area of the circle described by this equation is $\boxed{\frac{5\pi}{36}}$.

Solution 2: Alternatively, one can note that the resulting set of points is the inversion of the line $y = -2x-3$ with respect to the unit circle. The perpendicular line passing through the origin, $y = \frac{x}{2}$, intersects $y = -2x-3$ at $-\frac{6}{5} - \frac{3}{5}i$, which has a magnitude of $\frac{3}{\sqrt{5}}$, so its inversion in the unit circle has a magnitude of $\frac{\sqrt{5}}{3}$. This is the diameter of the resulting circle, so we get an area of $\boxed{\frac{5\pi}{36}}$.

Solution 3: To obtain a different inversive solution, let $w = (1-2i)z$. Then, $w - \overline{w} = 6i$ so the set of all feasible w is parametrized by the line $x + 3i$. Hence, $\frac{1}{w}$ describes a circle centered at $\frac{1}{6}i$ and of radius $\frac{1}{6}$. The area encompassed by this circle is $\frac{\pi}{36}$. However, since $|w|^2 = \frac{1}{5}|z|^2$ as $|1-2i|^2 = 5$, it follows that the area encompassed by $\frac{1}{z}$ is precisely 5 times that of $\frac{1}{w}$. This once more gives $\boxed{\frac{5\pi}{36}}$.

7. Consider a sequence $F_0 = 2$, $F_1 = 3$ that has the property $F_{n+1}F_{n-1} - F_n^2 = (-1)^n \cdot 2$. If each term of the sequence can be written in the form $a \cdot r_1^n + b \cdot r_2^n$, what is the positive difference between r_1 and r_2?

Answer: $\frac{\sqrt{17}}{2}$

Solution: Listing out the first few terms of the sequence, we have $F_0 = 2, F_1 = 3, F_2 = \frac{7}{2}, F_3 = \frac{19}{4}, F_4 = \frac{47}{8}$. Note that the terms of the sequence satisfy the recursive relation $F_{n+1} = \frac{F_n}{2} + F_{n-1}$. We will prove this inductively. Suppose that we already know that the property given in the problem and the recursive relation are satisfied for all F_n with $n \leq k$. Then, we want to show that if $F_{k+1} = \frac{F_k}{2} + F_{k-1}$ then $F_{k+1}F_{k-1} - F_k^2 = (-1)^k \cdot 2$. We have $F_{k+1}F_{k-1} - F_k^2 = \frac{F_kF_{k-1}}{2} + F_{k-1}^2 - F_k^2$. Note that $F_kF_{k-2} - F_{k-1}^2 = (-1)^{n-1} \cdot 2 \Rightarrow F_{k-1}^2 = F_kF_{k-2} + (-1)^n \cdot 2$. So,

$$\begin{aligned}\frac{F_kF_{k-1}}{2} + F_{k-1}^2 - F_k^2 &= \frac{F_kF_{k-1}}{2} + F_kF_{k-2} + (-1)^n \cdot 2 - F_k^2 \\ &= F_k\left(\frac{F_{k-1}}{2} + F_{k-2} - F_k\right) + (-1)^n \cdot 2 \\ &= F_k \cdot 0 + (-1)^n \cdot 2,\end{aligned}$$

which proves our claim. Now we know that the characteristic equation of the recurrence is $x^2 = \frac{x}{2} + 1$, and solving for x we get $x = \frac{1 \pm \sqrt{17}}{4}$. These are the values of r_1 and r_2, so their positive difference is $\boxed{\frac{\sqrt{17}}{2}}$.

8. If x and y are real numbers, compute the minimum possible value of

$$\frac{4xy(3x^2 + 10xy + 6y^2)}{x^4 + 4y^4}.$$

Answer: −1

Solution: Note that $x^4 + 4y^4 = (x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy)$ through the Sophie Germain identity. Also, the numerator can be written as

$$\begin{aligned}4xy(3x^2 + 10xy + 6y^2) &= 12x^3y + 40x^2y^2 + 24xy^3 \\ &= 5x^4 + 12x^3y + 40x^2y^2 + 24xy^3 + 20y^4 - 5(x^4 + 4y^4) \\ &= (x^2 + 2y^2 - 2xy)^2 + 4(x^2 + 2y^2 + 2xy)^2 - 5(x^4 + 4y^4).\end{aligned}$$

Then, we can decompose the given fraction as

$$\frac{(x^2 + 2y^2 - 2xy)^2 + 4(x^2 + 2y^2 + 2xy)^2}{(x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy)} - 5 = \frac{(x^2 + 2y^2 - 2xy)}{(x^2 + 2y^2 + 2xy)} + \frac{4(x^2 + 2y^2 + 2xy)}{(x^2 + 2y^2 - 2xy)} - 5.$$

By the AM-GM inequality, we have

$$\frac{(x^2 + 2y^2 - 2xy)}{(x^2 + 2y^2 + 2xy)} + \frac{4(x^2 + 2y^2 + 2xy)}{(x^2 + 2y^2 - 2xy)} \geq 4,$$

so the minimum possible value of the original expression is $\boxed{-1}$.

9. Let x, y, z be nonzero numbers, not necessarily real, such that

$$(x - y)^2 + (y - z)^2 + (z - x)^2 = 24yz$$

and

$$\frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} = 3.$$

Compute $\frac{x^2}{yz}$.

Answer: 5

Solution: Via factoring, we get

$$\frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} = 3$$

implies

$$(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = 0$$

or

$$(x + y + z)((x - y)^2 + (y - z)^2 + (z - x)^2) = 24(x + y + z)yz = 0.$$

As $y, z \neq 0$, we have $x + y + z = 0$ or $x = -y - z$. Then

$$\frac{x^2}{yz} = \frac{(-y-z)^2}{yz} = \frac{y^2 + 2yz + z^2}{yz} = \frac{y}{z} + \frac{z}{y} + 2.$$

Now, substituting $-y - z$ for x in the first equation gives us

$$\begin{aligned}(-y-z-y)^2 + (y-z)^2 + (z+y+z)^2 &= 4y^2 + 4yz + z^2 + y^2 - 2yz + z^2 + y^2 + 4yz + 4z^2 \\ &= 6y^2 + 6z^2 + 6yz \\ &= 24yz,\end{aligned}$$

or

$$\left(\frac{y}{z}\right)^2 - 3\left(\frac{y}{z}\right) + 1 = 0.$$

By the Quadratic Formula, we have

$$\frac{y}{z} = \frac{3 \pm \sqrt{5}}{2}.$$

It follows that the answer is

$$\begin{aligned}\frac{y}{z} + \frac{z}{y} + 2 &= \frac{3-\sqrt{5}}{2} + \frac{3+\sqrt{5}}{2} + 2 \\ &= \boxed{5}.\end{aligned}$$

10. Suppose that $p(x), q(x)$ are monic polynomials with nonnegative integer coefficients such that

$$\frac{1}{5x} \geq \frac{1}{q(x)} - \frac{1}{p(x)} \geq \frac{1}{3x^2}$$

for all integers $x \geq 2$. Compute the minimum possible value of $p(1) \cdot q(1)$.

Answer: 3

Solution: Rearranging the right side, we have that $3x^2(p(x) - q(x)) \geq p(x)q(x)$. By degree matching, it must be the case that $\deg p \geq \deg q$ and $\deg q \leq 2$.

Suppose first that $\deg q = 1$: that is, $q(x) = x + k$ for some k. Then, we need

$$\frac{1}{10} \geq \frac{1}{k+2} - \frac{1}{p(2)} \geq \frac{1}{12}.$$

$p(x)$ must also be linear, as otherwise $5x$ would eclipse $q(x)$ for large x.

Then, we are looking to minimize $(1+k)(1+\ell)$ such that $\frac{1}{10} \geq \frac{1}{k+2} - \frac{1}{\ell+2} \geq \frac{1}{12}$. Fortunately, in this particular case minimizing $(1+k)(1+\ell)$ turns out to be equivalent to minimizing k. To see this, fix k. As $\ell > k$ increases, there is a contiguous (possibly empty) range of ℓ such that $\frac{1}{10} \geq \frac{1}{k+2} - \frac{1}{\ell+2} \geq \frac{1}{12}$. Furthermore, as k increases, the start point of this range also increases.

So, since $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ minimizes k and ℓ, this gives $p(1)q(1) = 6$.

Now, suppose that $\deg q = 2$ and recall the condition $q(x)p(x) \leq 3x^2(p(x) - q(x))$. If $\deg p = 2$ as well, by monicity the right-hand side would have a degree of at most 3, impossible. So, $\deg p \geq 3$.

Since $p(1), q(1)$ are exactly the sums of coefficients of p, q it must imply that to beat the linear case we need a small number of coefficients.

First, we dispose of cases when $q(x) = x^2$. Indeed, note that at $x = 2$, $\frac{1}{4} - \frac{1}{8} > \frac{1}{10}$ so any $p(x)$ with leading coefficient at least x^3 cannot work. Next, look at $p(x) = x^k$. By some casework, we see that $k = 3$ leads to no solutions at $x = 2$ (as $\frac{1}{5} - \frac{1}{8} < \frac{1}{12}$) and similarly for $k = 4$ (as $\frac{1}{6} - \frac{1}{16} > \frac{1}{10}$ and $\frac{1}{7} - \frac{1}{16} < \frac{1}{12}$). However, at $k = 5$ we find the solution $q(x) = x^2 + 2x$. This yields $p(1)q(1) = 3$. If $k > 5$ then $\frac{1}{p(2)} \le \frac{1}{64}$ so $q(2)$ must increase. However, increasing $q(2)$ must increase either the x or constant coefficient. Hence, $p(x) = x^5$ is optimal.

If both q and p have a non-leading term, then $p(1)q(1) \ge 4$ so our answer must indeed be $\boxed{3}$.

Finally, we will check that $\frac{1}{5x} \ge \frac{1}{x^2+2x} - \frac{1}{x^5} \ge \frac{1}{3x^2}$ to verify the correctness of our solution. Since for all $x \ge 3$ we have $x^2 + 2x \ge 5x$, the left side must be satisfied.

For the right side, note that for all $x \ge 2$, $\frac{1}{x^5} \le \frac{1}{2(x^2+2x)}$. This follows as $2x \le x^2$ so $2(x^2+2x) \le 4x^2 \le x^4 \le x^5$. Furthermore, for all $x \ge 3$, $2(x^2+1) \le 3x^2$ (since rearranging yields $x^2 \ge 2$). Therefore,

$$\frac{1}{x^2+1} - \frac{1}{x^3+1} \ge \frac{1}{2(x^2+1)} \ge \frac{1}{3x^2}$$

as desired.