1. An ant starts at the point $(1, 1)$. It can travel along the integer lattice, only moving in the positive x and y directions. What is the number of ways it can reach $(5, 5)$ without passing through $(3, 3)$?

   Answer: 34

   Solution: In total, the ant must travel 8 units, meaning there are $\binom{8}{4}$ ways for it to reach $(5, 5)$, ignoring the missing point. Likewise, there are $\binom{4}{2}$ ways to reach the missing point, and $\binom{4}{2}$ ways to travel from the missing point to the end. Thus, there are $\binom{8}{4} - \binom{4}{2}\binom{4}{2} = 34$ ways.

2. Call a three-digit number $\overline{ABC}$ *spicy* if it satisfies $\overline{ABC} = A^3 + B^3 + C^3$. Compute the unique n for which both n and $n + 1$ are *spicy*.

   Answer: 370

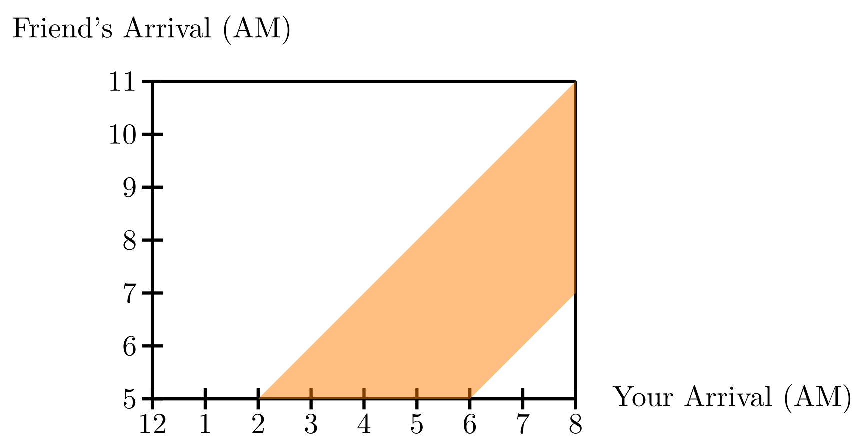
   Solution: We have two cases to consider: either $n + 1 = \overline{AB(C + 1)}$ or not. In the first case, notice that we must have $C = 0$. So, we have then that $100A + 10B = A^3 + B^3$. So, $A^3 + B^3$ is divisible by 10. Notice that these pair up: in other words, we must have that $A + B = 10$. With this in mind, we can compute $1^3 + 9^3 = 730, 2^3 + 8^3 = 520, 3^3 + 7^3 = 370, 4^3 + 6^3 = 280, 5^3 + 5^3 = 250$. Of these, $A = 3, B = 7, C = 0$ satisfies the conditions, yielding an answer of $\boxed{370}$.

   All that remains is to show that if C carries, then we cannot have both n and $n + 1$ spicy. Indeed, to have carry, we need $C = 9$. This implies that $B < 9$, else $B^3 + C^3 \geq 1000$. So, $n + 1 = \overline{A(B + 1)0}$. Therefore, $n = A^3 + B^3 + 729$ and $n + 1 = A^3 + (B + 1)^3$. Hence, $B^3 + 729 = (B + 1)^3$, impossible for $B \leq 9$ as $(B + 1)^3 - B^3 = 3B^2 + 3B + 1 \leq 7B^2 \leq 567$. Therefore, we have shown our answer is unique.

3. Every day you go to the music practice rooms at a random time from 12AM to 8AM and practice for 3 hours, while your friend goes at a random time from 5AM to 11AM and practices for 1 hour (the block of practice time need not be contained in the given time range for the arrival). What is the probability that you and your friend meet on at least 2 days in a given span of 5 days?

   Answer: $\frac{131}{243}$

   Solution:



   For a given day, the probability that you and your friend meet is $\frac{1}{3}$, which we can find by graphing the possible pairs of your and your friend's arrival times (see diagram). You can arrive no earlier than 3 hours before your friend and no later than 1 hour after your friend in order to meet. Then, the probability that you and your friend don't meet or only meet once in a span of 5 days is $\left(\frac{2}{3}\right)^5 + 5\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^4 = \frac{112}{243}$. Then, the probability that you meet at least twice is $1 - \frac{112}{243} = \frac{131}{243}$.

4. Frank mistakenly believes that the number 1011 is prime and for some integer x writes down $(x+1)^{1011} \equiv x^{1011} + 1 \pmod{1011}$. However, it turns out that for Frank's choice of x, this statement is actually true. If x is positive and less than 1011, what is the sum of the possible values of x?

   Answer: 3030

   Solution: We first find that $1011 = 3 \cdot 337$. By checking each of the values 0, 1, and 2, we can see that there are no restraints on $x \pmod 3$. Then, we want to find x such that

$$(x+1)^{1011} - x^{1011} - 1 \equiv 0 \pmod{337}.$$

   Using Fermat's little theorem, we can reduce this to

$$\begin{aligned}(x+1)^3 - x^3 - 1 &\equiv 0 \pmod{337}\\ \Rightarrow 3x^2 + 3x &\equiv 0 \pmod{337}\\ \Rightarrow 3x(x+1) &\equiv 0 \pmod{337}.\end{aligned}$$

   So, 337 divides x or 337 divides $x+1$. Therefore, our possibilities for x are 336, 337, 673, 674, and 1010. These values sum to $\boxed{3030}$.

5. A classroom has 30 seats arranged into 5 rows of 6 seats. Thirty students of distinct heights come to class every day, each sitting in a random seat. The teacher stands in front of all the rows, and if any student seated in front of you (in the same column) is taller than you, then the teacher cannot notice that you are playing games on your phone. What is the expected number of students who can safely play games on their phone?

   Answer: $\mathbf{\frac{163}{10}}$

   Solution: Let S be the total number of safe students. For each seat in row i and column j, let $S_{i,j}$ be the indicator that the person sitting there can play on their smartphone. In particular, $S_{i,j}$ is 1 if the student in seat (i,j) is safe, and 0 otherwise. Then, we can write $S = \sum_{i=1}^{5}\sum_{j=1}^{6} S_{i,j}$ by summing over each student's seat.

   By Linearity of Expectation, we have the the expectation of S is exactly the sum of the expectations of the $S_{i,j}$.

   So, we compute each of these expectations and sum them to get our answer. Let $p_{i,j}$ be the probability that the student in seat (i,j) is safe: that is, $p_{i,j}$ is the probability that $S_{i,j} = 1$. Then, the expectation of $S_{i,j}$ is $p_{i,j}$.

   To compute $p_{i,j}$ we make the following observation: a student in row i can only not safely play games on their phone if they are taller than all the students in front of them: that is, in slots $(1,j),(2,j),\ldots,(i-1,j)$. This occurs with probability $\frac{1}{i}$: by symmetry, the tallest of these i students (the $i-1$ before them and themself) is equally likely to be in any of the i positions. So, $p_{i,j} = 1 - \frac{1}{i}$.

   So, our total sum and answer is

$$\sum_{i=1}^{5}\sum_{j=1}^{6} 1 - \frac{1}{i} = 30 - \sum_{i=1}^{5}\sum_{j=1}^{6}\frac{1}{i} = 30 - 6\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) = \boxed{\frac{163}{10}}.$$

6. Let $\mathcal{A}$ be the set of finite sequences of positive integers $a_1, a_2, \ldots, a_k$ such that $|a_n - a_{n-1}| = a_{n-2}$ for all $3 \leqslant n \leqslant k$. If $a_1 = a_2 = 1$, and $k = 18$, determine the number of elements of $\mathcal{A}$.

Answer: 1597

Solution 1: Define a set $\mathcal{S}$ of sequences $(s_1, s_2, \ldots, s_{k-2})$, where $s_i = 1$ if $a_{i+2} - a_{i+1} > 0$ and $s_i = -1$ otherwise. It is clear that there exists a bijection between $\mathcal{S}$ and $\mathcal{A}$.

We note that no sequence in $\mathcal{S}$ can ever contain two -1s in a row. This is due to the fact that if $a_{j+2} - a_{j+1} < 0$ for some j, then we must have $a_{j+3} = a_{j+2} + a_{j+1} > a_{j+2}$. It then suffices to count the number of sequences $\mathcal{S}$ such that $s_1 = 1$ (because $a_3 = 2$) and contains no two -1s that are next to each other. Let our sequence have j -1s. Then we may place $j-1$ 1s in between each pair of consecutive -1s. Then we may still place $15 - (j + j - 1) = 16 - 2j$ 1s, which by stars and bars can occur in $\binom{16-j}{j}$ ways. Thus our answer is

$$\sum_{j=0}^{8} \binom{16-j}{j} = \boxed{1597}.$$

Solution 2: Let $\mathcal{B}$ be a sequence $(b_1, b_2, \ldots, b_k)$ such that b_n represents the number of finite sequences in the set $\mathcal{A}_n = a_1, a_2, \ldots, a_n$. Now note that moving from b_{n-1} to b_n, a particular sequence will 'split' if $a_{n-1} > a_{n-2}$, in which case one of the resulting sequences will be able to be split in the next element of b and the other won't. Additionally, if a particular sequence doesn't split in this element of $\mathcal{B}$, it must split in the next. This behavior combines together to form a fibonacci series, where beginning with b_4, $b_n = b_{n-1} + b_{n-2}$, as in any given sequence there are b_{n-2} splits from b_{n-1}.
This leads to the series $b_2 = 1, b_3 = 1, b_4 = 2, b_5 = 3, b_6 = 6, \ldots, b_{16} = 610, b_{17} = 987, b_{18} = \boxed{1597}$.

7. Let n_0 be the product of the first 25 primes. Now, choose a random divisor n_1 of n_0, where a choice n_1 is taken with probability proportional to $\varphi(n_1)$. ($\varphi(m)$ is the number of integers less than m which are relatively prime to m.) Given this n_1, we let n_2 be a random divisor of n_1, again chosen with probability proportional to $\varphi(n_2)$. Compute the probability that $n_2 \equiv 0 \bmod 2310$.

Answer: $\frac{256}{5929}$

Solution: First, we show that $\sum_{d|n} \varphi(d) = n$. This is true for all n, but we'll only show it for squarefree n for simplicity. We proceed by induction (all divisors of a squarefree n are themselves squarefree). For any prime p, $\varphi(p) + \varphi(1) = p$.

Now, suppose that $n = p \cdot n'$. Then, we may pair up divisors of n via $(d, p \cdot d)$ with $d \mid n'$. So,

$$\sum_{d|n} \varphi(d) = \sum_{d|n'} \varphi(d) + \varphi(p \cdot d) = \sum_{d|n'} (1 + (p-1))\varphi(d) = pn' = n$$

as desired. We used that $\varphi(ab) = \varphi(a) \cdot \varphi(b)$ if $\gcd(a, b) = 1$. As a note, to prove the general case, we strengthen the statement to prime powers and strengthen pairs to tuples of divisors.

This means that each choice of n_1 actually happens with probability $\frac{\varphi(n_1)}{n_0}$, and each choice of n_2 happens with probability $\frac{\varphi(n_2)}{n_1}$ given a specific n_1.

To begin, let's compute the probability that $n_1 \equiv 0 \bmod 2310$. Let $m = \frac{n_0}{2310}$. We will need that φ is a multiplicative function: that is, if $\gcd(m, n) = 1$ then $\varphi(mn) = \varphi(m)\varphi(n)$.

Then, the probability of getting $n_1 = 2310d$ for d a specific divisor of m is exactly

$$\frac{\varphi(2310d)}{n_0} = \frac{\varphi(2310)\varphi(d)}{2310m} = \frac{\varphi(2310)}{2310} \cdot \frac{\varphi(d)}{m}.$$

So, summing over all choices of d means that the probability that 2310 divides n_1 is

$$\sum_{d|m} \frac{\varphi(2310)}{2310} \cdot \frac{\varphi(d)}{m} = \frac{\varphi(2310)}{2310}.$$

We've finished one step, so what remains is to repeat it. In particular, note that we didn't care much about what n_0 was: all that mattered was that n_0 was squarefree and $n \equiv 0 \bmod 2310$. So, conditioned on $n_1 \equiv 0 \bmod 2310$, it follows that the probability that $n_2 \equiv 0 \bmod 2310$ is also $\frac{\varphi(2310)}{2310}$.

Therefore, our answer is

$$\begin{aligned}
(\text{probability } n_2 \equiv 0 \bmod 2310) =& (\text{probability } n_1 \equiv 0 \bmod 2310) \\
& \cdot (\text{probability } n_2 \equiv 0 \bmod 2310 \text{ given } n_1 \equiv 0 \bmod 2310) \\
=& \left(\frac{\varphi(2310)}{2310}\right)^2 \\
=& \left(\frac{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11}\right)^2 \\
=& \left(\frac{16}{77}\right)^2 \\
=& \boxed{\frac{256}{5929}}
\end{aligned}$$

8. Given that $20^{22} + 1$ has exactly 4 prime divisors $p_1 < p_2 < p_3 < p_4$, determine $p_1 + p_2$.

Answer: 490

Solution: Suppose that p divides $20^{22} + 1$. Then, since $20^{22} \equiv -1 \bmod p$ it must follow that $20^{44} \equiv 1 \bmod p$. Let k be the minimal positive integer such that $20^k \equiv 1 \bmod p$. Then, as $k \mid 44$ and $k \nmid 22$, it follows that either $k = 44$ or $k = 4$.

As $20^{p-1} \equiv 1 \bmod p$ for any p by Fermat's Little Theorem, the next claim is that $44 \mid p - 1$ if we are in the first case. Indeed, suppose not. Then, write $p - 1 = 44k + r$, with $0 < r < 44$. We have that $20^r \equiv 1 \bmod p$, but this is impossible as $k = 44$ is minimal.

We may test to see that $20^{22} \equiv -1 \bmod 89$: since $89 \cdot 23 = 2047$ and $89 \cdot 7 = 623$ we have $2^{44} \equiv 2048^4 \equiv 1 \bmod 89$ and $5^{22} \equiv 25 \cdot (625)^5 \equiv 100 \cdot 8 \equiv 88 \bmod 89$. Hence, $20^{22} \equiv -1 \bmod 89$ as desired.

Now, suppose that $20^4 \equiv 1 \bmod p$. This implies that $20^4 - 1 = 3 \cdot 7 \cdot 19 \cdot 401 \equiv 0 \bmod p$. Note that $3, 7, 19 = \pm 1 \bmod 20$ so $20^{22} + 1 \equiv 2 \text{mod} \text{any of them}$. However, we may check that $20^{22} = (-1)^{11} \equiv -1 \bmod 401$.

So, to finish off, we show that there are no other $44 \mid p - 1$ with $p < 401$ and $20^{22} \equiv -1 \bmod p$. By divisibility considerations, we end up with having to check that $20^{22} \not\equiv -1 \bmod \{353, 397\}$.

Let's handle 397 first. Note that $20^{22} = 3^{11} = 243 \cdot 243 \cdot 3 = 154^2 \cdot 3 \not\equiv -1 \bmod 397$. One way of showing the last inequivalence is noting that $3^{-1} = \frac{2 \cdot 397 - 1}{3} = 265$ and showing that $154^2 \not\equiv 132 \bmod 397$.

Now we deal with 353. Note that $2^{44} = 2^4 \cdot (-35)^4 = 70^4 = 4900^2 = 42^2 \equiv -1 \bmod 353$.

Hence, it suffices to show that $5^{22} \not\equiv 1 \bmod 353$. We may write $5^{22} - 1 = (5^{11} - 1)(5^{11} + 1)$, but since $5^{11} \equiv (-81)^2 \cdot 125 \equiv 5 \cdot 405^2 \equiv 5 \cdot 52^2 \not\equiv 1, -1 \bmod 353$, we are done.

Therefore, our answer is indeed $89 + 401 = \boxed{490}$.

9. For any positive integer n, let $f(n)$ be the maximum number of groups formed by a total of n people such that the following holds: every group consists of an even number of members, and every two groups share an odd number of members. Compute $\sum_{n=1}^{2022} f(n) \bmod 1000$.

Answer: 242

Solution: For simplicity, we may translate the problem into the following form. For positive integer n, $f(n)$ is the maximum number m for which one can find a collection of m subsets $C_1, \ldots, C_m \subseteq \{1, \ldots, n\}$ such that C_i contains an even number of elements for $1 \le i \le n$, and $C_i \cap C_j$ contains an odd number of elements for $1 \le i \ne j \le n$.

We prove that $f(2k) = f(2k-1) = 2k-1$ for any positive integer k. Clearly $f(2k) \ge f(2k-1)$ because any collection of subsets of $\{1, \ldots, 2k-1\}$ is also a collection of subsets of $\{1, \ldots, 2k\}$. Hence it suffices to show that $f(2k-1) \ge 2k-1$, and $f(2k) \le 2k-1$.

To show that $f(2k-1) \ge 2k-1$, it suffices to construct $2k-1$ subsets such that the conditions hold. Let $C_i = \{i, 2k-1\}$ for $1 \le i \le 2k-2$ and $C_{2k-1} = \{1, \ldots, 2k-2\}$. These subsets satisfying the conditions, so $f(2k-1) \ge 2k-1$.

It remains to show that $f(2k) \le 2k-1$. Let $n = 2k$. Suppose otherwise that there exists n subsets $C_1, \ldots, C_n$ of $\{1, \ldots, n\}$ such that C_i contains an even number of elements for $1 \le i \le n$, and $C_i \cap C_j$ contains an odd number of elements for $1 \le i \ne j \le n$. We will show a contradiction from here.

We start with a few notations. For a finite set X, let $|X|$ be the number of elements in X. For finite sets X, Y, let $X \Delta Y$ be the symmetric difference $(X \cup Y) \setminus (X \cap Y)$. We have the following properties regarding symmetric difference Δ.

- Δ is commutative and associative. Therefore we may write $\Delta_{i=1}^{t} X_i := X_1 \Delta \cdots \Delta X_t$.
- For sets X and Y, $X \Delta Y = \emptyset$ if and only if $X = Y$.
- $|X \Delta Y| \equiv |X| + |Y| \pmod 2$ for any finite sets X, Y.
- $X \cap (Y \Delta Z) = (X \cap Y) \Delta (X \cap Z)$ for any finite sets X, Y, Z.

Now we can continue our argument for showing a contradiction. First we show that there exists a nonempty collection of the subsets $\{C_i\}_{i \in I}$ where $I \ne \emptyset$ such that $\Delta_{i \in I} C_i = \emptyset$. For any $I \subseteq \{1, \ldots, n\}$, note that

$$|\Delta_{i \in I} C_i| \equiv \sum_{i \in I} |C_i| \equiv 0 \pmod 2.$$

Hence there are a total of 2^n choices for I, while there are only 2^{n-1} possibilities for $\Delta_{i \in I} C_i$. Hence by the pigeonhole principle, there exists distinct subsets I_1, I_2 such that $\Delta_{i \in I_1} C_i = \Delta_{i \in I_2} C_i$. Then we have for $I = I_1 \Delta I_2 \ne \emptyset$,

$$\Delta_{i \in I} C_i = \Delta_{i \in I_1 \Delta I_2} C_i = (\Delta_{i \in I_1} C_i) \, \Delta \, (\Delta_{i \in I_2} C_i) = \emptyset.$$

This shows the existence of the nonempty collection indexed by I. Now we argue that $|I|$ is odd. Fix $j \in I$. We have

$$0 = |\emptyset| = |C_j \cap \Delta_{i\in I} C_i| = |\Delta_{i\in I} C_j \cap C_i| \equiv |C_j| + \sum_{i\in I, i\neq j} |C_j \cap C_i| \equiv |I| - 1 \pmod 2.$$

Because n is even and $|I|$ is odd, it follows that we can find $j \in \{1, \dots, n\} \setminus I$. We repeat the same argument as above and get

$$0 = |\emptyset| = |C_j \cap \Delta_{i\in I} C_i| = |\Delta_{i\in I} C_j \cap C_i| \equiv \sum_{i\in I} |C_j \cap C_i| \equiv |I| \pmod 2.$$

This shows that $|I|$ is even. However, $|I|$ cannot be both odd and even, so we have a contradiction. Therefore we have $f(2k) = f(2k-1) = 2k-1$. Hence

$$\sum_{n=1}^{2022} f(n) = \sum_{k=1}^{1011} (f(2k) + f(2k-1)) = 2\sum_{k=1}^{1011} (2k-1) = 2 \times 1011^2 = 2044242.$$

It follows that the last 3 digits are $\boxed{242}$.

10. How many solutions are there to the equation

$$x^2 + 2y^2 + z^2 = xyz$$

where $1 \le x, y, z \le 200$ are positive even integers?

Answer: 13

Solution: We begin by observing that $(4, 4, 4)$ is a valid solution to our equation. Now, assume that (a, b, c) is a valid solution to our equation, where a, b, c are all even. This means that a is a solution to the polynomial $t^2 - (bc)t + 2b^2 + c^2$. By Vieta's formulas, the other solution to this polynomial is $bc - a$, or equivalently $\frac{2b^2+c^2}{a}$. Since the latter is a positive even integer, so is the former (since they are equal). Therefore, we can conclude that if (a, b, c) is a solution to the equation, then so is $(bc - a, b, c)$. Similarly, if (a, b, c) is a solution, then so is $(a, b, ab - c)$.

Now treating the equation as the polynomial $2t^2 - (ac)t + a^2 + c^2$, which has root b, we know that the other root, $\frac{ac}{2} - b$, or equivalently $\frac{a^2+c^2}{2b}$, is a positive even integer and hence gives us another valid solution. Thus, if (a, b, c) is a solution, then so is $\left(a, \frac{ac}{2} - b, c\right)$.

Finally, it suffices to show that *all* such solutions can be found by beginning at the solution $(4, 4, 4)$ and then performing these operations to "jump" to other solutions. Let (a, b, c) be any valid solution. We claim that one of the operations $(a, b, c) \to (bc-a, b, c)$, $(a, b, c) \to (a, b, ab-c)$, or $(a, b, c) \to \left(a, \frac{ac}{2} - b, c\right)$ will decrease the sum of the three values, *unless* $(a, b, c) = (4, 4, 4)$.

If the operation $(a, b, c) \to \left(a, \frac{ac}{2} - b, c\right)$ decreases the sum of the variables, then we are done. Otherwise, we must have $b \le \frac{ac}{2} - b$, or in other words $b \le \frac{ac}{4}$. This is also equivalent to $4b^2 \le abc = a^2 + 2b^2 + c^2$ or $2b^2 \le a^2 + c^2$. This means that $b \le \max\{a, c\}$.

Without loss of generality, assume that $a \le c$. Then we claim the operation $(a, b, c) \to (a, b, ab - c)$ decreases the sum of the three values unless $(a, b, c) = (4, 4, 4)$. If didn't then we have $ab-c \ge c$ or $ab \ge 2c$. This can be rewritten as $abc \ge 2c^2$ or $a^2 + 2b^2 \ge c^2$. However, this means that $c^2 \le 3\max\{a, b\}^2$. We have 2 cases we must consider:

Case 1: $c \ge b \ge a$. In this case, we have $abc = a^2 + 2b^2 + c^2 \le \max\{a, b\}^2 + 2\max\{a, b\}^2 + 3\max\{a, b\}^2 = 6b^2$. In other words, $ac \le 6b$ or $a \le 6$ since $c \ge b$. Actually, this inequality is strict since equality only holds if $a = b$ (for the first inequality to be an equality) and $c = b$ (for the second inequality to be an equality). However, if $a = b = c$ the it is easy to see the only solution is $(4, 4, 4)$. This forces $a = 4$. Plugging this into the original equation and solving for b, we get $b = c - \frac{1}{2}\sqrt{2c^2 - 32}$. However, in order for our sum to increase, we must have $ab > 2c$, or $2b > c$ because $a = 4$. This means that we must have $2c - \sqrt{2c^2 - 32} > c$ or $c < \sqrt{32}$. This forces $c = 4$ and subsequently $b = 4$.

Case 2: $c \geq a \geq b$. In this case, we have $abc = a^2 + 2b^2 + c^2 < \max\{a, b\}^2 + 2\max\{a, b\}^2 + 3\max\{a, b\}^2 = 6a^2$. In other words, we have $bc \leq 6a$, or $b \leq 6$ since $c \geq a$. Again, we can make this inequality strict by applying the same argument as in Case 1. This forces $b = 4$. Plugging this into our original equation and solving for a, we get $a = 2c - \sqrt{3c^2 - 32}$. However, we also know that in order for our sum to increase, we must have $ab = 4a > 2c$, or $8c - 4\sqrt{3c^2 - 32} > 2c$, which means $3c^2 < 128$ or $c < \sqrt{128/3}$. It is easily verified that $c = 6$ does not produce any solutions, which means $c = 4$ and subsequently $a = 4$.

We do not have to consider the third case $b \geq c \geq a$ since $b \leq \max\{a, c\}$.

Thus, for any solution (a, b, c) we can repeatedly apply some operation until the sum no longer decreases. However, by casework, we have shown that the only time this occurs is when $(a, b, c) = (4, 4, 4)$. Therefore, to produce all solutions, we simply start from $(4, 4, 4)$ and repeatedly apply the operations until one of the values exceeds 200. Doing this manually is not difficult and produces the solutions (4, 4, 4), (4, 4, 12), (4, 20, 12), (4, 20, 68), (4, 116, 68), (12, 4, 4), (12, 4, 44), (12, 20, 4), (44, 4, 12), (44, 4, 164), (68, 20, 4), (68, 116, 4), (164, 4, 44). The answer is thus $\boxed{13}$.