Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise.
No calculators.

1. An ant starts at the point $(1, 1)$. It can travel along the integer lattice, only moving in the positive x and y directions. What is the number of ways it can reach $(5, 5)$ without passing through $(3, 3)$?

2. Call a three-digit number $\overline{ABC}$ *spicy* if it satisfies $\overline{ABC} = A^3 + B^3 + C^3$. Compute the unique n for which both n and $n + 1$ are *spicy*.

3. Every day you go to the music practice rooms at a random time from 12AM to 8AM and practice for 3 hours, while your friend goes at a random time from 5AM to 11AM and practices for 1 hour (the block of practice time need not be contained in the given time range for the arrival). What is the probability that you and your friend meet on at least 2 days in a given span of 5 days?

4. Frank mistakenly believes that the number 1011 is prime and for some integer x writes down $(x + 1)^{1011} \equiv x^{1011} + 1 \pmod{1011}$. However, it turns out that for Frank's choice of x, this statement is actually true. If x is positive and less than 1011, what is the sum of the possible values of x?

5. A classroom has 30 seats arranged into 5 rows of 6 seats. Thirty students of distinct heights come to class every day, each sitting in a random seat. The teacher stands in front of all the rows, and if any student seated in front of you (in the same column) is taller than you, then the teacher cannot notice that you are playing games on your phone. What is the expected number of students who can safely play games on their phone?

6. Let $\mathcal{A}$ be the set of finite sequences of positive integers $a_1, a_2, \ldots, a_k$ such that $|a_n - a_{n-1}| = a_{n-2}$ for all $3 \leqslant n \leqslant k$. If $a_1 = a_2 = 1$, and $k = 18$, determine the number of elements of $\mathcal{A}$.

7. Let n_0 be the product of the first 25 primes. Now, choose a random divisor n_1 of n_0, where a choice n_1 is taken with probability proportional to $\varphi(n_1)$. ($\varphi(m)$ is the number of integers less than m which are relatively prime to m.) Given this n_1, we let n_2 be a random divisor of n_1, again chosen with probability proportional to $\varphi(n_2)$. Compute the probability that $n_2 \equiv 0 \bmod 2310$.

8. Given that $20^{22} + 1$ has exactly 4 prime divisors $p_1 < p_2 < p_3 < p_4$, determine $p_1 + p_2$.

9. For any positive integer n, let $f(n)$ be the maximum number of groups formed by a total of n people such that the following holds: every group consists of an even number of members, and every two groups share an odd number of members. Compute $\sum_{n=1}^{2022} f(n) \bmod 1000$.

10. How many solutions are there to the equation
$$x^2 + 2y^2 + z^2 = xyz$$
where $1 \leq x, y, z \leq 200$ are positive even integers?