1. Compute

$$\int_0^{10} (x-5)+(x-5)^2+(x-5)^3 dx.$$

Answer: $\mathbf{\frac{250}{3}}$

Solution: This integral is equivalent to

$\int_{-5}^{5} x+x^2+x^3dx = \int_{-5}^{5} x^2dx = \frac{5^3}{3} - \frac{(-5)^3}{3} = \boxed{\frac{250}{3}}$.

2. Water is flowing out through the smaller base of a hollow conical frustum formed by taking a downwards pointing cone of radius 12m and slicing off the tip of the cone in a cut parallel to the base so that the radius of the cross-section of the slice is 6m (meaning the smaller base has a radius of 6m). The height of the frustum is 10m. If the height of the water level in the frustum is decreasing at 3m/s and the current height is 5m, then the volume of the water in the frustum is decreasing at d m^3/s. Compute d.

Answer: 243

Solution: Let the radii of the bases of the frustum be $r_1 = 6$ and $r_2 = 12$, and let the height be $h = 10$. Also, let the current height of the water in the frustum be $h_c = 5$, the radius of the surface of the water be r_c, and the height of the part of the cone that was cut off to make the frustum be h_0.

First, we can find h_0 using similar triangles. We have

$$\frac{h_0}{h_0+h} = \frac{r_1}{r_2} \Rightarrow h_0r_2 = h_0r_1 + hr_1 \Rightarrow h_0 = \frac{hr_1}{r_2-r_1} = \frac{10\cdot 6}{12-6} = 10.$$

We can also find r_c in terms of h_c:

$$\frac{r_c}{r_1} = \frac{h_0+h_c}{h_0} \Rightarrow r_c = r_1\frac{h_0+h_c}{h_0}.$$

The volume of the water in the frustum is

$$V = \frac{1}{3}\pi r_c^2(h_0+h_c) - \frac{1}{3}\pi r_1^2 h_0 = \frac{1}{3}\pi r_1^2\frac{(h_0+h_c)^2}{h_0^2}(h_0+h_c) - \frac{1}{3}\pi r_1^2 h_0.$$

Taking the derivative with respect to time gives

$$\begin{aligned}\frac{dV}{dt} &= \frac{1}{3}\pi r_1^2 \cdot 3\frac{(h_0+h_c)^2}{h_0^2}\cdot\frac{dh_c}{dt}\\ &= \frac{1}{3}\pi\cdot 6^2\cdot 3\cdot\frac{(10+5)^2}{10^2}\cdot 3\\ &= \boxed{243\pi}.\end{aligned}$$

3. Compute the value of

$$\int_{-\pi}^{\pi} \frac{e^{x^2}-e^{-x^2}}{e^{x^2}-x\sqrt{2}}\,|x|\,dx.$$

Answer: $\mathbf{\frac{1}{2}\ln\left(e^{2\pi^2}-2\pi^2\right)}$

Solution: Let I denote the value of the integral. We first split the integral into two parts

$$I = \int_{-\pi}^{0} \frac{e^{x^2} - e^{-x^2}}{e^{x^2} - x\sqrt{2}} |x| \, dx + \int_{0}^{\pi} \frac{e^{x^2} - e^{-x^2}}{e^{x^2} - x\sqrt{2}} |x| \, dx.$$

Since $|x| = x$ for $x > 0$ and $|x| = -x$ for $x < 0$, we have

$$I = \int_{-\pi}^{0} \frac{e^{x^2} - e^{-x^2}}{e^{x^2} - x\sqrt{2}} (-x) dx + \int_{0}^{\pi} \frac{e^{x^2} - e^{-x^2}}{e^{x^2} - x\sqrt{2}} x dx.$$

Performing a substitution $x \to -x$ on the first integral gives

$$I = \int_{0}^{\pi} \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + x\sqrt{2}} x dx + \int_{0}^{\pi} \frac{e^{x^2} - e^{-x^2}}{e^{x^2} - x\sqrt{2}} x dx = \int_{0}^{\pi} \frac{\left(2e^{2x^2} - 2\right) x}{e^{2x^2} - 2x^2} dx$$

Now, the indefinite integral can be solved by performing the substitution $v = e^{2x^2} - 2x^2$.

$$\int \frac{\left(2e^{2x^2} - 2\right) x}{e^{2x^2} - 2x^2} dx = \int \frac{dv}{2v} = \frac{1}{2} \ln v.$$

Therefore, $\int_0^\pi \frac{\left(2e^{2x^2} - 2\right) x}{e^{2x^2} - 2x^2} dx = \left[\frac{1}{2} \ln v\right]\Big|_0^\pi = \left[\frac{1}{2} \ln \left(e^{2x^2} - 2x^2\right)\right]\Big|_0^\pi = \boxed{\frac{1}{2} \ln \left(e^{2\pi^2} - 2\pi^2\right)}.$