1. If $f(x) = x^4 + 4x^3 + 7x^2 + 6x + 2022$, compute $f'(3)$.

   Answer: 264

   Solution: $f'(x) = 4x^3 + 12x^2 + 14x + 6$. Substitute 3 into x gives $f'(3) = 264$. **Solution 2:** It's easy to recognize that $f(x) = (1+x)^4 + (1+x)^2 + 2020$. So $f'(x) = 4(1+x)^3 + 2(1+x)$. Substitute 3 gives the correct answer.

2. The straight line $y = ax + 16$ intersects the graph of $y = x^3$ at 2 distinct points. What is the value of a?

   Answer: 12

   Solution: The solution is $\boxed{12}$.

   For there to be 2 distinct intersections, we must have that our straight line is tangent to the graph.

   Let the tangent point be (x_0, y_0). Then, $ax_0 + 16 = y_0 = x_0^3$ (point of intersection) and $3x_0^2 = a$ (point of tangency). We have $3x_0^2 \cdot x_0 + 16 = x_0^3$. Solving this gives $x_0 = -2$, which in turn gives $a = 12$.

   We can check relatively easily that this value of a works.

3. For $k = 1, 2, \ldots$, let f_k be the number of times

$$\sin\left(\frac{k\pi x}{2}\right)$$

   attains its maximum value on the interval $x \in [0, 1]$. Compute

$$\lim_{k\to\infty} \frac{f_k}{k}.$$

   Answer: $\frac{1}{4}$

   Solution: The period of f_k is $4/k$. The maximum is attained in the first quarter of each period, so the number of times the maximum is attained is equal to the number of periods there are in $[0, 1]$, rounded up. Thus,

$$f_k = \left\lceil \frac{1}{4/k} \right\rceil = \left\lceil \frac{k}{4} \right\rceil.$$

   Then,

$$\left| \frac{f_k}{k} - \frac{1}{4} \right| \le \frac{1}{k} \to 0$$

   as $k \to \infty$. Thus, the answer is $\boxed{\frac{1}{4}}$.

4. Evaluate the integral:

$$\int_{\frac{\pi^2}{4}}^{4\pi^2} \sin(\sqrt{x})dx.$$

   Answer: $-4\pi - 2$.

Solution: We begin by applying a u-substitution: $u = \sqrt{x}$. This gives us

$$\int_{\frac{\pi}{2}}^{2\pi} 2u\sin(u)du.$$

Next, we integrate by parts ($u = u, dv = \sin(u)du$.) This yields:

$$2(u\cos(u) + \sin(u))$$

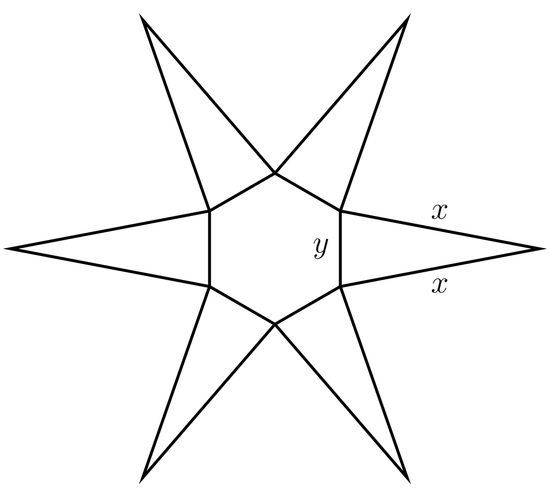
evaluated from 2π to $\frac{\pi}{2}$. Now, we plug in the boundaries, which leads us to the answer, $-4\pi - 2$.

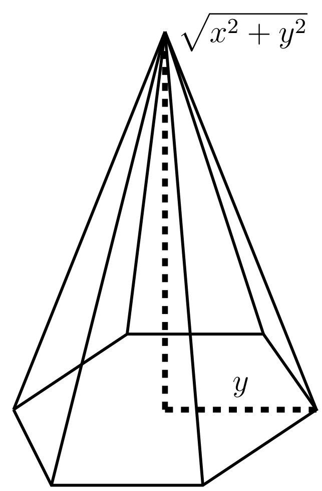
5. A net for a hexagonal pyramid is constructed by placing a triangle with side lengths x, x, and y on each side of a regular hexagon with side length y. What is the maximum volume of the pyramid formed by the net if $x + y = 20$?

Answer: $\mathbf{128\sqrt{15}}$

Solution: The height of the pyramid can be calculated as $\sqrt{x^2 - y^2}$. Thus, the volume can be expressed as $V = 6 \cdot \frac{y^2\sqrt{3}}{4} \cdot \frac{1}{3}\sqrt{x^2 - y^2} = \frac{\sqrt{3}}{2} \cdot \left(y^2\sqrt{x^2 - y^2}\right)$. Substituting in $x = 20 - y$ we get $V = \frac{\sqrt{3}}{2} \cdot \left(y^2\sqrt{400 - 40y}\right)$. Deriving, we get $\frac{dV}{dy} = 2y\sqrt{400 - 40y} - \frac{20y^2}{\sqrt{400-40y}} = \frac{800y - 100y^2}{\sqrt{400-40y}}$. The maximum value will occur when $\frac{dV}{dy} = 0$, which occurs at $y = 8$. This means $x = 20 - 8 = 12$, and our volume $V = \frac{\sqrt{3}}{2} \cdot \left(8^2\sqrt{12^2 - 8^2}\right) = \boxed{128\sqrt{15}}$

Picture to illustrate result:





6. Let

$$f(x) = \cos\left(x^3 - 4x^2 + 5x - 2\right).$$

If we let $f^{(k)}$ denote the kth derivative of f, compute $f^{(10)}(1)$. For the sake of this problem, note that $10! = 3628800$.

Answer: 907200

Solution: Perform the substitution $u = x - 1$. Then our function becomes $g(u) = \cos\left(u^3 - u^2\right)$. The Taylor expansion of $g(u)$ can be written as

$$g(u) = 1 - \frac{(u^3 - u^2)^2}{2!} + \frac{(u^3 - u^2)^4}{4!} - \ldots$$

There is a term of the form u^{10} in $(u^3 - u^2)^4$. The coefficient of u^{10} can be written as $\frac{\binom{4}{2}}{4!} = 1/4$, which means our answer is $10!/4 = \boxed{907200}$.

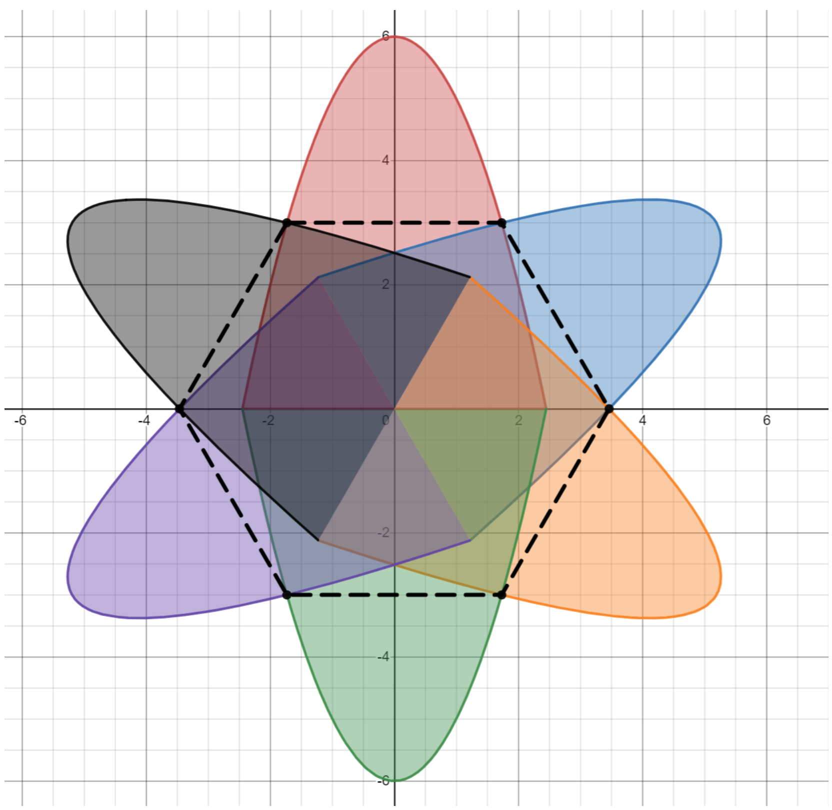
7. Let

$$A_j = \left\{(x, y) : 0 \le x \sin\left(\frac{j\pi}{3}\right) + y \cos\left(\frac{j\pi}{3}\right) \le 6 - \left(x \cos\left(\frac{j\pi}{3}\right) - y \sin\left(\frac{j\pi}{3}\right)\right)^2\right\}$$

The area of $\cup_{j=0}^{5} A_j$ can be expressed as $m\sqrt{n}$. What is the area?

Answer: $42\sqrt{3}$

Solution: Consider the case of $j = 0$; then our inequality simply becomes $0 \le y \le 6 - x^2$, which is the tip of a parabola with vertex $(0, 6)$ and opens downward. Then, note that A_j is just A_0 rotated by $\pi/3$ about the origin j times, so graphing produces the following shape, which can be broken down into a central hexagon and six parabolic portions:



The key step is to find where the corners of the hexagon are. It will be very difficult to find the intersection of A_0 and A_1 directly, so instead we note that by symmetry, the intersection happens along the line $\pi/3$ above the x-axis. Along that line, $y = x\sqrt{3}$, so the problem becomes solving

$$x\sqrt{3} = 6 - x^2$$

which gives $x = \sqrt{3}$. Then, the side length of the hexagon is $2\sqrt{3}$; the area of the hexagon is then $18\sqrt{3}$.

Next, the parabolic parts. Since there are six identical parts, we can simply find the area of A_0 not in the hexagon and multiply by six. This area is

$$6\int_{-=\sqrt{3}}^{\sqrt{3}} 6-x^2-3dx = 12\left[3x - x^3/3\right]_{-\sqrt{3}}^{\sqrt{3}} = 12(3\sqrt{3}-\sqrt{3}) = 24\sqrt{3}.$$

Adding back the area of the hexagon gives the total area to be $18\sqrt{3}+24\sqrt{3} = 42\sqrt{3}$.

8. Given that

$$A = \sum_{n=1}^{\infty} \frac{\sin(n)}{n},$$

determine $\lfloor 100A \rfloor$.

Answer: 107

Solution: From Euler's Formula, this sum is clearly

$$\text{Im}\left(\sum_{n=1}^{\infty} \frac{e^{in}}{n}\right)$$

Now recall the Taylor Expansion of $-\log(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$. Hence, it suffices to find

$$\text{Im}(-\log(1-e^i))$$

We have that $e^i = \cos(1) + i\sin(1)$ (angles taken in radians). Now we convert $1 - e^i = 1 - \cos(1) - i\sin(1)$ into an exponential form. It has magnitude

$$\sqrt{(1-\cos(1))^2 + \sin^2(1)} = \sqrt{2-2\cos(1)}$$

and argument $-\arctan\frac{\sin(1)}{1-\cos(1)}$. Therefore,

$$1 - e^i = \sqrt{2-2\cos(1)} \cdot e^{-i\tan^{-1}\frac{\sin(1)}{1-\cos(1)}}$$

Taking the log of this we have

$$\frac{1}{2}\log(2-2\cos(1)) - i\tan^{-1}\frac{\sin(1)}{1-\cos(1)}$$

so our desired answer is

$$\tan^{-1}\frac{\sin 1}{1-\cos 1} = \tan^{-1}\frac{\sin(\pi-1)}{1+\cos(\pi-1)} = \tan^{-1}\tan\left(\frac{1}{2}(\pi-1)\right)$$

and so our desired answer is $\lfloor 50(\pi-1) \rfloor \approx 50(2.14) = \boxed{107}$

9. Let $f(x,y) = (\cos x + y\sin x)^2$. We may express $\max_x f(x,y)$, the maximum value of $f(x,y)$ over all values of x for a given fixed value of y, as a function of y, call it $g(y)$. Let the smallest positive value x which achieves this maximum value of $f(x,y)$ for a given y be $h(y)$. Compute

$$\int_1^{2+\sqrt{3}} \frac{h(y)}{g(y)}\,dy.$$

Answer: $\frac{\pi^2}{18}$

Solution: Let's look at $a\cos x + b\sin x$ for general $a, b \in \mathbb{R}$ with $b \neq 0$ instead first. We claim that this is equal to $\sqrt{a^2+b^2}\cos\big(x - \tan^{-1}(\frac{b}{a})\big)$. Indeed, let $\theta = \tan^{-1}(\frac{b}{a})$. Then, we may write

$$\begin{aligned} a\cos x + b\sin x &= \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos x + \frac{b}{\sqrt{a^2+b^2}}\sin x\right) \\ &= \sqrt{a^2+b^2}(\cos\theta\cos x + \sin\theta\sin x) \\ &= \sqrt{a^2+b^2}\cos(x-\theta) \\ &= \sqrt{a^2+b^2}\cos\left(x - \tan^{-1}(\frac{b}{a})\right) \end{aligned}$$

So, as a consequence, we find that $g(y) = 1 + y^2$, as cosine can only take values from 0 to 1, and that $h(y) = \tan^{-1}(y)$. Notice that $\frac{\mathrm{d}}{\mathrm{d}y}(h(y))^2 = 2h(y)h'(y) = \frac{2\tan^{-1}(y)}{1+y^2} = 2\frac{h(y)}{g(y)}$. Therefore, we have that

$$\int_1^{2+\sqrt{3}} \frac{h(y))}{g(y))}\,\mathrm{d}y = \frac{1}{2}(\tan^{-1}(y))^2\Big|_{y=1}^{2+\sqrt{3}} = \frac{1}{2}(\tan^{-1}(2+\sqrt{3}))^2 - \frac{1}{2}(\tan^{-1}(1))^2.$$

Now as a last step, we need to determine $\theta = \tan^{-1}(2+\sqrt{3})$. One way to do this is to compute that $\sin\theta = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$, so by the half angle formula, $\sin(2\theta) = -\frac{\sqrt{3}}{2}$. Therefore, $2\theta = \cdot\frac{5\pi}{6}$ and our desired angle is $\frac{5\pi}{12}$. So, our final answer is

$$\frac{1}{2}\cdot\frac{25\pi^2}{144} - \frac{1}{2}\cdot\frac{\pi^2}{16} = \boxed{\frac{\pi^2}{18}}$$

10. Consider the set of continuous functions f, whose n^{th} derivative exists for all positive integer n, satisfying $f(x) = \frac{\mathrm{d}^3}{\mathrm{d}x^3}f(x)$, $f(0) + f'(0) + f''(0) = 0$, and $f(0) = f'(0)$. For each such function f, let $m(f)$ be the smallest nonnegative x satisfying $f(x) = 0$. Compute all possible values of $m(f)$.

Answer: **0,** $\frac{5\pi\sqrt{3}}{9}$

Solution: Solution sketch: express $f(x)$ as Taylor series around $x = 0$: $f(x) = \sum_{i=0}^{\infty} a_i\frac{x^i}{i!}$ and notice that the third derivative condition implies that we may rewrite this as

$$f(x) = f(0)\sum_{i=0}^{\infty}\frac{x^{3i}}{(3i)!} + f'(0)\sum_{i=0}^{\infty}\frac{x^{3i+1}}{(3i+1)!} + f''(0)\sum_{i=0}^{\infty}\frac{x^{3i+2}}{(3i+2)!}.$$

Now, we can use a roots of unity filter on e^x: in particular, if $\omega = e^{\frac{2\pi i}{3}}$ is a primitive third root of unity, then each of the sums can be written as $\frac{e^x + \alpha e^{\omega x} + \beta e^{\omega^2 x}}{3}$ for appropriate α and β. Therefore, we obtain

$$f(x) = \frac{f(0) + f'(0) + f''(0)}{3}\cdot e^x + \gamma_1 e^{\omega x} + \gamma_2 e^{\omega^2 x}$$

where γ_1, γ_2 depend on the values at 0. The first initial condition therefore nulls out the e^x term.

Now noticing that $e^{\omega x} = e^{-\frac{x}{2}} \cdot \left[\cos\left(\frac{\sqrt{3}}{2}x\right) + i\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$ and $e^{\omega^2 x}$ flips the last term to a $-$ yields that we may write

$$f(x) = e^{-\frac{x}{2}}\left[\lambda\cos\left(\frac{\sqrt{3}}{2}x\right) + \mu\sin\left(\frac{\sqrt{3}}{2}x\right)\right].$$

Plugging in $x = 0$ yields $f(0) = \lambda$, and taking a derivative gives $f'(0) = -\frac{\lambda}{2} + \frac{\mu\sqrt{3}}{2}$. With $f(0) = f'(0)$, we have that $\mu = \sqrt{3}f(0) = \lambda\sqrt{3}$.

So, suppose that $f(0) \neq 0$. Then, if $f(x) = 0$, we have $\tan\left(\frac{\sqrt{3}}{2}x\right) = -\frac{\lambda}{\mu} = -\frac{1}{\sqrt{3}}$. So, the smallest positive $\frac{\sqrt{3}}{2}x$ is $\pi - \frac{\pi}{6}$ (tan repeats every π) and hence the minimum value is $\frac{5\pi\sqrt{3}}{9}$.

Combining with the possibility of $f(0) = 0$ yields our answer of $\boxed{0, \frac{5\pi\sqrt{3}}{9}}$.