1. For some positive integer n, $2021 - 2(5^n)$ can be expressed as the sum and difference of distinct integer powers of 5. Compute 5^n.

   Answer: 625

   Solution: Since $2021 = 31041_5 = 31101_5 - 10_5$, it follows that $2021 = 3(5^4) + 5^3 + 5^2 - 5^1 + 5^0$. Therefore, $2021 - 2(5^4) = 5^4 + 5^3 + 5^2 - 5^1 + 5^0$ can be expressed as the sum and difference of distinct powers of 5. Therefore, $5^n = 5^4 = \boxed{625}$.

2. Find the smallest integer $n \geq 2021$ such that $30n^3 + 143n^2 + 117n - 56$ is divisible by 13.

   Answer: 2024

   Solution 1: Since $30n^3 + 143n^2 + 117n - 56 \equiv 4n^3 + 9 \mod 13$, it follows that it is divisible by 13 exactly when $n^3 \equiv 1 \mod 13$. Since 2 is a primitive root of 13, $n^3 \equiv 1 \mod 13$ when $n \equiv 2^4, 2^8, 2^{12} \mod 13$. Therefore, $30n^3 + 143n^2 + 117 - 56$ is divisible by 13 if and only if $n \equiv 3, 9, 1 \mod 13$. Since $2021 \equiv 6 \mod 13$, the smallest value of n is $\boxed{2024} \equiv 9 \mod 13$.

   Solution 2: Factoring,

   $$30n^3 + 143n^2 + 117n - 56 = (2n + 7)(3n - 1)(5n + 8).$$

   Therefore, the expression is divisible by 13 if and only if $2n + 7$, $3n - 1$, or $5n + 8$ is congruent to $0 \mod 13$. Solving for each of these, we get that the expression is divisible by 13 if and only if $n \equiv 3, 9, 1 \mod 13$. Since $2021 \equiv 6 \mod 13$, the smallest value of n is $\boxed{2024} \equiv 9 \mod 13$.

3. Suppose that a positive integer n has 6 positive divisors where the 3^{rd} smallest is a and the a^{th} smallest is $\frac{n}{3}$. Find the sum of all possible value(s) of n.

   Answer: 120

   Solution: Since n has 6 divisors, either $n = p^5$ for some prime p or $n = p^2q$ for some distinct primes p and q. Moreover, since $\frac{n}{3}$ is a divisor of n, it follows that 3 must be a divisor of n. However, since the only divisors of n that can be greater than $\frac{n}{3}$ are $\frac{n}{2}$ and n, it follows that a must be equal to 4 or 5. Since $a \neq 3$ is also a divisor of n, it follows that $n = p^2q$ for some distinct primes p and q so the only possible values of n are 12, 45, and 75. Of these values, we see that only $n = 45$ and $n = 75$ satisfy the conditions:

   $$1, 2, 3, 4, 6, 12$$
   $$1, 3, 5, 9, 15, 45$$
   $$1, 3, 5, 15, 25, 75$$

   Therefore, the sum of all possible values of n is $45 + 75 = \boxed{120}$.

4. A positive integer n has 4 positive divisors such that the sum of its divisors is $\sigma(n) = 2112$. Given that the number of positive integers less than and relative prime to n is $\phi(n) = 1932$, find the sum of the proper divisors of n.

   Answer: 91

   Solution 1: Since n has four divisors, either $n = p^3$ for some prime p or $n = pq$ for some distinct primes p and q. Suppose that $n = p^3$ for some prime p. Then

   $$11^3 = 1131 < 1932 = \phi(n) < n = p^3 < \sigma(n) = 2112 < 2197 = 13^3$$

   implies that $11 < p < 13$, which cannot be true, so $n \neq p^3$ for any prime p.

Therefore, $n = pq$ for some distinct primes p and q. In this case, we have that $\sigma(n) = (p+1)(q+1) = pq + (p+q) + 1$ and $\phi(n) = (p-1)(q-1) = pq - (p+q) + 1$. Therefore, the sum of the proper divisors of n is equal to

$$p + q + 1 = \frac{\sigma(n) - \phi(n)}{2} + 1 = \frac{2112 - 1932}{2} + 1 = \boxed{91}.$$

Solution 2: Observe that $n = 2021 = 43(47)$. The sum of the proper divisors of n is $1+43+47 = \boxed{91}$.

5. $15380 - n^2$ is a perfect square for exactly four distinct positive integers. Given that $13^2 + 37^2 = 1538$, compute the sum of these four possible values of n.

   Answer: 300

   Solution: Observe that for any c that $(x+cy)^2+(cx-y)^2 = (y+cx)^2+(cy-x)^2 = (c^2+1)(x^2+y^2)$. Letting $c = 3$ and $(x, y) = (13, 37)$, we have that $(13 + 3 \cdot 37)^2 + (3 \cdot 13 - 37)^2 = (37 + 3 \cdot 13)^2 + (3 \cdot 37 - 13)^2 = (3^2 + 1)(13^2 + 37^2) = 15380$. Therefore, the sum of the possible values of n is $|x + cy| + |cx - y| + |y + cx| + |cy - x|$. Since all of these values are positive, the sum is equal to $2c(x + y) = 2(3)(13 + 37) = \boxed{300}$.

   To double-check the values of n, doing the arithmetic yields that $2^2 + 124^2 = 76^2 + 98^2 = 15380$ and $2 + 124 + 76 + 98 = 300$.

6. Find the sum of all possible values of abc where a, b, c are positive integers that satisfy

$$a = \gcd(b, c) + 3,$$
$$b = \gcd(a, c) + 3,$$
$$c = \gcd(a, b) + 3.$$

   Answer: 436

   Solution: First, note that since the gcd of any two positive integers is at least 1, it follows that $a, b, c \geq 4$. Without loss of generality, let $a \geq b \geq c \geq 4$. Then $a = \gcd(b, c) + 3 \leq c + 3$ can be at most $c + 3$. We now perform casework on the value of a:

   i) If $a = c$, then $a = b = c$, so $a = \gcd(b, c) + 3 = a + 3$ which is a contradiction.

   ii) If $a = c + 1$, then $b = \gcd(a, c) + 3 = 4$ so $c = 4$ and $a = c + 1 = 5$. However, this is a contradiction as $a = 5 \neq 7 = \gcd(b, c) + 3$.

   iii) If $a = c + 2$ and a is odd, then $b = \gcd(a, c) + 3 = 4$ so $c = 4$ and $a = c + 2 = 6$. However, this contradicts the assumption that a is odd. On the other hand, if a is even, then $b = \gcd(a, c) + 3 = 5$. Since $b = 5$ while a and c are both even, $c = 4$ and $a = 6$. However, this is a contradiction as $a = 6 \neq 4 = \gcd(b, c) + 3$.

   iv) If $a = c + 3$, then $b = 4$ or $b = 6$. If $b = 4$, then $c = 4$ and $a = c + 3 = 7$ which gives the solution $(a, b, c) = (7, 4, 4)$. If $b = 6$ then $\gcd(b, c)+3 = a \geq b = 6$ so $\gcd(b, c) \geq 3$ so the only possible value of c in this case is 6. This gives us the only other solution $(a, b, c) = (9, 6, 6)$.

   Therefore, since our only solutions are $(a, b, c) = (7, 4, 4)$ and $(a, b, c) = (9, 6, 6)$ (up to rearrangement), the sum of all possible values of abc is $112 + 324 = \boxed{436}$.

7. Let a be the positive integer that satisfies the equation

$$1+\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+...+\frac{29}{30}=\frac{a}{30!}.$$

What is the remainder when a is divided by 17?

Answer: 3

Solution: We note that

$$a=30!+\frac{30!}{2}+\frac{2\cdot 30!}{3}+...+\frac{29\cdot 30!}{30}.$$

Since all of these terms in the sum are divisible by 17 except $\frac{16\cdot 30!}{17}$, we have that

$$\begin{aligned} a &\equiv \frac{16\cdot 30!}{17} \mod 17 \\ &\equiv 16\cdot(16!)\cdot(13)! \mod 17 \end{aligned}$$

Wilson's theorem gives us that $16\cdot(16!)\equiv 1 \mod 17$ so $a\equiv 13! \mod 17$. It also tells us that $15!\equiv 1 \mod 17$, so $a\equiv 13!\equiv \frac{1}{14\cdot 15}\equiv 11\cdot 8\equiv \boxed{3} \mod 17$.

8. Compute the remainder when

$$2018^{2019^{2020}}+2019^{2020^{2021}}+2020^{2020^{2020}}+2021^{2020^{2019}}+2022^{2021^{2020}}$$

is divided by 2020.

Answer: 2

Solution: Using binomial expansion, we have that

$$2018^{2019^{2020}}=(-2)^{2019^{2020}}+2019^{2020}\times(-2)^{2019^{2020}-1}\times 2020+...$$

$$2019^{2020^{2021}}=1-2020^{2021}\times 2020+...$$

$$2021^{2020^{2019}}=1+2020^{2019}\times 2020+...$$

$$2022^{2021^{2020}}=(2)^{2021^{2020}}+2021^{2020}\times(2)^{2021^{2020}-1}\times 2020+...$$

where the ... are divisible by higher powers of 2020. We note that $\varphi(2020)=\varphi(4)\varphi(5)\varphi(101)=800$. Then $\varphi(800)=\varphi(32)\varphi(25)=80$. Then $2020\equiv 20 \pmod{80}$. Now, we consider

$$\begin{aligned} 2019^{20} &\equiv 1-20\times 2020+\binom{20}{2}\times 2020^2 \pmod{800} \\ &= 1-400 \pmod{800} \end{aligned}$$

Similarly, we consider

$$\begin{aligned} 2021^{2020} &\equiv 1+20\times 2020+\binom{20}{2}\times 2020^2 \pmod{800} \\ &= 1+400 \pmod{800} \end{aligned}$$

Then we know that

$$(-2)^{2019^{2020}} \equiv -1 \times (2)^{1-400} \equiv (-2) \times (2)^{-400} \pmod{2020}$$

and

$$(2)^{2021^{2020}} \equiv 2 \times 2^{400} \pmod{2020}$$

Then we know that $2^{400} \equiv 2^{-400} \pmod{2020}$ since $400 = 800/2$. So, these cancel out and the total remainder is $\boxed{2}$ mod 2020.

9. Find the least positive integer k such that there exists a set of k distinct positive integers $\{n_1, n_2, \ldots, n_k\}$ that satisfy the equation

$$\prod_{i=1}^{k} \left(1 - \frac{1}{n_i}\right) = \frac{72}{2021}.$$

Answer: 28

Solution: Suppose that a set $\{n_1, n_2, \ldots, n_k\}$ satisfies the given equation. Without loss of generality, let $n_1 < n_2 < \ldots < n_k$. Moreover, $n_1 \neq 1$ as $1 - \frac{1}{1} = 0$. Therefore, $n_i \geq i + 1$ for $i \in \{1, 2, \ldots, k\}$. Hence, we have that

$$\frac{72}{2021} = \prod_{i=1}^{k} \left(1 - \frac{1}{n_i}\right) \geq \prod_{i=1}^{k} \left(1 - \frac{1}{i+1}\right) = \prod_{i=1}^{k} \frac{i}{i+1} = \frac{1}{k+1}.$$

Rearranging, we have that $k + 1 \geq \frac{2021}{72} > 28$ so $k \geq 28$.

Now consider the 28-element set $\{2, 3, \ldots, 23, 25, 26, 27, 28, 43, 47\}$. Since

$$\left(\frac{1}{2}\right) \cdots \left(\frac{22}{23}\right) \left(\frac{24}{25}\right) \cdots \left(\frac{27}{28}\right) \left(\frac{42}{43}\right) \left(\frac{46}{47}\right) = \left(\frac{1}{23}\right) \left(\frac{24}{28}\right) \left(\frac{42}{43}\right) \left(\frac{46}{47}\right) = \frac{72}{2021},$$

there exists a satisfactory set for $k = \boxed{28}$.

10. Compute the smallest positive integer n such that $n^{44}+1$ has at least three distinct prime factors less than 44.

Answer: 161

Solution: For any prime p to divide $n^{44} + 1$, it must be that $n^{44} \equiv -1 \mod p$, which implies that $n^{88} \equiv 1 \mod p$. Therefore, for any prime $p > 2$, $-1 \not\equiv 1 \mod p$ so $ord_p(n) \nmid 44$. Similarly, for any prime p, it follows that $ord_p(n) | 88$. Together, this implies that $8 | ord_p(n)$ for any prime $p > 2$. However, since $ord_p(n) | p - 1$, it follows that $8 | p - 1$ or equivalently, that $p \equiv 1 \mod 8$. Since the only primes less than 44 that satisfy this condition are 17 and 41, three distinct prime factors must be 2, 17, and 41.

For $p = 2$, it follows that $n^{44} + 1 \equiv -1 \mod 2$ exactly when $n \equiv 1 \mod 2$, which is the same as n being odd.

For any primitive root g of modulo $p = 17$, it follows that $n^{44} + 1 \equiv 0 \mod 17$ exactly when n mod 17 is equivalent to either g^2, g^6, g^{10}, or g^{14}. Since 2 is a solution as $2^{44} + 1 = (2^4)^{11} + 1 \equiv (-1)^{11} + 1 \equiv 0 \mod 17$, we know there is (at least) one g such that $g^2 \equiv 2 \mod 17$. Substituting for g^2, it follows that $n \mod 17$ must be equivalent to either 2, 2^3, -2, or -2^3 so $n \equiv 2, 8, 9, 15$ mod 17.

Similarly, for any primitive root g of $p = 41$, it follows that $n^{44} + 1 \equiv 0 \mod 41$ exactly when $n \mod 41$ is equivalent to either g^5, g^{15}, g^{25}, or g^{35}. Since 3 is a solution as $3^{44} + 1 = (3^4)^{11} + 1 \equiv (-1)^{11} + 1 \equiv 0 \mod 41$, we know there is (at least) one g such that $g^5 \equiv 3 \mod 41$. Substituting for g^5, it follows that $n \mod 41$ must be equivalent to either 3, 3^3, -3, or -3^3 so $n \equiv 3, 14, 27, 38 \mod 41$.

By CRT, we get the following table for each of the possible cases in mod 17 and 41:

mod 697	2 mod 17	8 mod 17	9 mod 17	15 mod 17
3 mod 41	495	331	536	372
14 mod 41	342	178	383	219
27 mod 41	478	314	519	355
38 mod 41	325	161	366	202

Therefore, the smallest solution is the smallest odd number in the table $n = \boxed{161}$.