1. A rectangular pool has diagonal 17 *units* and area 120 $units^2$. Joey and Rachel start on opposite sides of the pool when Rachel starts chasing Joey. If Rachel runs 5 units/sec faster than Joey, how long does it take for her to catch him?

   Answer: 23/5

   Solution: We can solve that this pool $(l+w)^2 = (l^2+w^2)+2lw = 529 = 23^2$. So, Rachel catches Joey in $\boxed{23/5}$ seconds.

2. Alice plays a game with her standard deck of 52 cards. She gives all of the cards number values where Aces are 1's, royal cards are 10's and all other cards are assigned their face value. Every turn she flips over the top card from her deck and creates a new pile. If the flipped card has value v, she places $12-v$ cards on top of the flipped card. For example: if she flips the 3 of diamonds then she places 9 cards on top. Alice continues creating piles until she can no longer create a new pile. If the number of leftover cards is 4 and there are 5 piles, what is the sum of the flipped over cards?

   Answer: 17

   Solution: We note that if v is the value of the flipped card in a pile, then the total number of cards in the pile is $13-v$. Hence, if $v_1, ..., v_5$ are the flipped values, there are $65-(v_1+v_2+v_3+v_4+v_5)$ cards in piles. With the extra 4 cards leftover, this must sum to 52. Thus, $v_1+v_2+v_3+v_4+v_5 = \boxed{17}$.

3. There are 5 people standing at $(0,0)$, $(3,0)$, $(0,3)$, $(-3,0)$, and $(-3,0)$ on a coordinate grid at a time $t=0$ seconds. Each second, every person on the grid moves exactly 1 unit up, down, left, or right. The person at the origin is infected with covid-19, and if someone who is not infected is at the same lattice point as a person who is infected, at any point in time, they will be infected from that point in time onwards. (Note that this means that if two people run into each other at a non-lattice point, such as $(0,1.5)$, they will not infect each other.) What is the maximum possible number of infected people after $t=7$ seconds?

   Answer: 0

   Solution: For this solution, we will define the distance between two people as the length of the shortest path between them, where a path can only travel along grid lines. For a given pair of people, we can see that any motion one of the two people makes will either increase or decrease the path length by 1 unit; there is no way for one person to move and keep the path length constant. Since both people are moving at the same time, this means that at each time step, their options are to increase their path length by 2 units, keep it the same length, or decrease it by 2 units. Thus, no matter what, the parity of the path length remains constant. This means that one person can only infect another if the initial path length between the two has the same parity as 0; that is, the path length is even. Since all people initially have a path length of 3 from the person at the origin, none of them can get infected.

4. Kara gives Kaylie a ring with a circular diamond inscribed in a gold hexagon. The diameter of the diamond is 2mm. If diamonds cost $\$100/mm^2$ and gold costs $\$50/mm^2$, what is the cost of the ring?

   Answer: $\$100\pi + \$75\sqrt{3}$

   Solution: We know that the area of a hexagon with side length 1 mm is $\frac{3\sqrt{3}}{2}mm^2$. So, we have $\$100\pi + \$50(\frac{3\sqrt{3}}{2} - \pi) = \boxed{\$50\pi + \$75\sqrt{3}}$.

5. Find the number of three-digit integers that contain at least one 0 or 5. The leading digit of the three-digit integer cannot be zero.

Answer: 388

Solution: First, consider the number of two digit numbers from 00 to 99 inclusive, that contain an 0 and/or a 5. There are 10 that have a 0 in the ones digit and 10 that have a zero in the tens digit. Of these, one number (00) falls into both categories. So, there are $10 + 10 - 1 = 19$ two-digit numbers that contain a 0. Similarly, there are 19 two-digit numbers that contain a 5. There are two numbers (05 and 50) that contain both a 0 and a 5. So, $19 + 19 - 2 = 36$ "two-digit" numbers contain a 0 and/or a 5.

A three-digit integer contains a 0 and/or a 5 if its hundreds digit is a 5 or its tens/ones digit contains a 0 and/or 5. There are $100 + 8 * (36) = \boxed{388}$.

6. What is the sum of the solutions to $\frac{x+8}{5x+7} = \frac{x+8}{7x+5}$

Answer: −7

Solution: We solve the system. Two ways: 1) $x + 8 = 0$, and 2)$5x + 7 = 7x + 5$. 1) $x + 8 = 0, x = -8$; 2) $5x + 7 = 7x + 5, x = 1$;

Therefore, the sum of the solutions is -7

7. Let BC be a diameter of a circle with center O and radius 4. Point A is on the circle such that $AOB = 45°$. Point D is on the circle such that line segment OD intersects line segment AC at E and OD bisects $\angle AOC$. Compute the area of ADE, which is enclosed by line segments AE and ED and minor arc $\widehat{AD}$.

Answer: $3\pi - 2\sqrt{2}$

Solution: Since OD bisects $\angle AOC$, we can compute the area of our desired region by subtracting the area of $\triangle AOC$ from the area of sector AOC and dividing the result by 2. Since $\angle AOB = 45°$, $\angle AOC = 135°$. It follows that the area of sector AOC is $\frac{135}{360} \cdot 4^2\pi = 6\pi$ and the area of $\triangle AOC$ is $\frac{1}{2} \cdot 4^2 \cdot \sin 135° = 4\sqrt{2}$. Hence, the area of ADE is $\frac{6\pi - 4\sqrt{2}}{2} = \boxed{3\pi - 2\sqrt{2}}$.

8. William is a bacteria farmer. He would like to give his fiancé 2021 bacteria as a wedding gift. Since he is an intelligent and frugal bacteria farmer, he would like to add the least amount of bacteria on his favourite infinite plane petri dish to produce those 2021 bacteria.

The infinite plane petri dish starts off empty and William can add as many bacteria as he wants each day. Each night, all the bacteria reproduce through binary fission, splitting into two. If he has infinite amount of time before his wedding day, how many bacteria should he add to the dish in total to use the least number of bacteria to accomplish his nuptial goals?

Answer: 8

Solution: Since we want the least amount of bacteria to be added to the petri dish, we can interpret each bacterium as powers of two since each one reproduces through binary fission for an indefinite amount of time and thus use a binary representation of 2021 and count the bits. Doing so, we get 11111100101. There are 8 bits, so the answer is 8.

9. The frozen yogurt machine outputs yogurt at a rate of 5 froyo^3/second. If the bowl is described by $z = x^2 + y^2$ and has height 5 froyos, how long does it take to fill the bowl with frozen yogurt?

Answer: $5\pi/2$ seconds

Solution: First we find the volume of the bowl. We see that z ranges from 0 to 5 froyos, so we have

$$V = \int_0^5 \pi z dz = \frac{25\pi}{2}$$

So, it takes $\frac{25\pi}{10} = \frac{5\pi}{2}$ seconds to fill the bowl.

10. Prankster Pete and Good Neighbor George visit a street of 2021 houses (each with individual mailboxes) on alternate nights, such that Prankster Pete visits on night 1 and Good Neighbor George visits on night 2, and so on. On each night n that Prankster Pete visits, he drops a packet of glitter in the mailbox of every n^{th} house. On each night m that Good Neighbor George visits, he checks the mailbox of every m^{th} house, and if there is a packet of glitter there, he takes it home and uses it to complete his art project. After the 2021^{th} night, Prankster Pete becomes enraged that none of the houses have yet checked their mail. He then picks three mailboxes at random and takes out a single packet of glitter to dump on George's head, but notices that all of the mailboxes he visited had an odd number of glitter packets before he took one. In how many ways could he have picked these three glitter packets? Assume that each of these three was from a different house, and that he can only visit houses in increasing numerical order.

Answer: 1540 ways

Solution: On every odd-numbered day n, Pete adds a packet to every mailbox whose label is a multiple of n. On every even numbered day m, George takes away a packet from every mailbox whose label is a multiple of m (and which has packets to take away). For each even-numbered house, each packet added on an odd day n can be mapped with a day $2n$ where George is taking away packets. So each even-numbered house ends up with 0 packets. Also, George never visits any odd-numbered houses, so the number of packets added in an odd-numbered house is the number of factors of that house number. Then, all the houses with an odd number of packets are odd squares. There are 22 odd squares less than 2021, and since Prankster Pete chooses three of those houses to pick a packet from, there are $\binom{22}{3} = \boxed{1540}$ ways for him to choose a packet on the final day.

11. The taxi-cab length of a line segment with endpoints (x_1, y_1) and (x_2, y_2) is $|x_1 - x_2| + |y_1 - y_2|$. Given a series of straight line segments connected head-to-tail, the taxi-cab length of this path is the sum of the taxi-cab lengths of its line segments. A goat is on a rope of taxi-cab length $\frac{7}{2}$ tied to the origin, and it can't enter the house, which is the three unit squares enclosed by $(-2, 0), (0, 0), (0, -2), (-1, -2), (-1, -1), (-2, -1)$. What is the area of the region the goat can reach? (Note: the rope can't "curve smoothly"—it must bend into several straight line segments.)

Answer: $\mathbf{\frac{167}{8}}$

Solution: This is similar to typical dog-on-a-leash problems but this time we must use the taxi-cab metric, not the normal Euclidean metric. The analog to a circle under the taxi-cab metric is a diamond. In quadrants I, II, and IV, we get the entire quarter-diamonds, which are right triangles with legs $\frac{7}{2}$. In quadrant III, we must avoid the house, which uses 2 units of the rope's length, leaving $\frac{3}{2}$ length. On both sides of the house, this sweeps out a quarter-diamond that is a right triangle with legs $\frac{3}{2}$. Reaching the very back of the house, we see that we have $\frac{3}{2} - 1 = \frac{1}{2}$ units of length left, allowing us to, on both sides of the house, sweep out a quarter-diamond that is a right triangle with legs $\frac{1}{2}$.

Summing these areas up, we get

$$2\left(\frac{1}{2}\left(\frac{1}{2}\right)^2+\frac{1}{2}\left(\frac{3}{2}\right)^2\right)+3\left(\frac{1}{2}\right)\left(\frac{7}{2}\right)^2=\boxed{\frac{167}{8}}.$$

There's a slightly nicer way to calculate this area after seeing what we sweep out: it's the entire diamond of radius $\frac{7}{2}$ minus the house and 5/8-ths of a unit square.

12. Parabola P, $y = ax^2 + c$ has $a > 0$ and $c < 0$. Circle C, which is centered at the origin and lies tangent to P at P's vertex, intersects P at only the vertex. What is the maximum value of a, possibly in terms of c?

    Answer: $-\frac{1}{2c}$

    Solution: The circle must have radius c and be centered at the origin, so its equation is $x^2 + y^2 = c^2$. Along with $y = ax^2 + c$, we want our only solution to be $(0, c)$. Substituting $y = ax^2 + c$, we have $x^2 + a^2x^4 + 2acx^2 + c^2 = c^2$, so $x^2(ax^2 + 2ac + 1) = 0$. Ignoring the $x^2 = 0$ solution, we have $x = \pm\sqrt{-\frac{2ac+1}{a^2}}$. We want this solution to be imaginary, so we want $2ac + 1 \geq 0$, which means $a \leq \frac{-1}{2c}$.

13. Emma has the five letters: A, B, C, D, E. How many ways can she rearrange the letters into words? Note that the order of words matter, ie ABC DE and DE ABC are different.

    Answer: 1920

    Solution: There are $5 * 4 * 3 * 2 * 1 = 120$ ways to rearrange the letters into one word.

    There are $4 * 5! = 480$ ways to rearrange the letters into two words.

    There are $6 * 5! = 720$ ways to rearrange the letters into three words.

    There are $4 * 5! = 480$ ways to rearrange the letters into four words.

    There are $1 * 5! = 120$ ways to rearrange the letters into five words.

    In total that is

    $$120 + 480 + 720 + 480 + 120 = 1920$$

    ways to rearrange the 5 distinct letters into words.

14. Seven students are doing a holiday gift exchange. Each student writes their name on a slip of paper and places it into a hat. Then, each student draws a name from the hat to determine who they will buy a gift for. What is the probability that no student draws himself/herself?

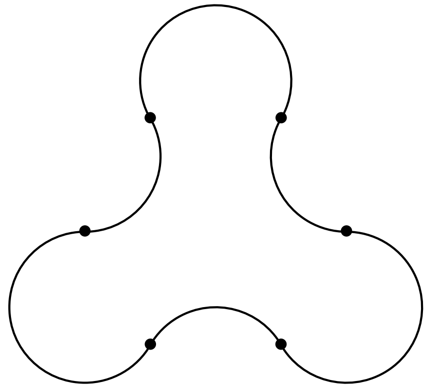
    Answer: $\frac{103}{280}$

    Solution: The number of ways such that each student cannot draw himself/herself is the number of derangements of a 7-element set. Let the number of derangements of a 7-element set be N. The formula for N is $7!\sum_{k=0}^{7}\frac{(-1)^k}{k!} = 1854$.

    In particular, the formula for the number of derangements can be seen explicitly using PIE. Let a fixed point be a person who draws their own name. There are $\binom{7}{1} * 6!$ ways to have one fixed point in the set of seven students. There are $\binom{7}{2} * 5!$ ways to have two fixed points in the set of seven students. Continuing in this fashion, there are $\binom{7}{7} * 1!$ ways to have seven fixed points in the set of seven students. Now using PIE, we see that there are $7! - \binom{7}{1} * 6! + \binom{7}{2} * 5! - \ldots - \binom{7}{7} * 0! = 7!\sum_{k=0}^{7}\frac{(-1)^k}{k!} = 1854$ ways to have no fixed points.

    Then, the probability of no fixed points is just $\frac{1854}{7!} = \frac{103}{280}$.

15. We model a fidget spinner as shown below (include diagram) with a series of arcs on circles of radii 1. What is the area swept out by the fidget spinner as it's turned $60°$?



Answer: $\frac{13}{2}\pi + 3\sqrt{3}$

Solution: Diagram is most useful in seeing the solution. The longest part of the fidget spinner will sweep out $60°$ of a circle of radius 3, and symmetry makes this happen three times for each long part of the fidget spinner. In these 3 sectors, we get 3 areas that can each be decomposed as a side length 2 equilateral triangle and 2 $120°$ parts of circles of radius 1. If we sum these, we get $\frac{1}{2}\pi 3^2 + 2\pi + 3\frac{2^2\sqrt{3}}{4} = \boxed{\frac{13}{2}\pi + 3\sqrt{3}}$.

16. Let a,b,c be the sides of a triangle such that

$$gcd(a, b) = 3528, gcd(b, c) = 1008, gcd(a, c) = 504$$

Find the value of $a * b * c$. Write your answer as a prime factorization.

Answer: $2^{12} * 3^9 * 7^5$

Solution: $a = 2^3 * 3^3 * 7^2, b = 2^5 * 3^2 * 7^2, c = 2^4 * 3^4 * 7^1$

17. Let the roots of the polynomial $f(x) = 3x^3 + 2x^2 + x + 8 = 0$ be p, q, and r. What is the sum $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$?

Answer: $\frac{-1}{8}$

Solution: First notice 0 is not a root of $f(x)$. Notice that since p, q, and r are the roots of the polynomial $f(x)$, $\frac{1}{p}$, $\frac{1}{q}$, and $\frac{1}{r}$ are the roots of the polynomial $g(x) = x^3 * f(\frac{1}{x})$. ($f(\frac{1}{x}) = \frac{3}{x^3} + \frac{2}{x^2} + \frac{1}{x} + 8$ is not a polynomial). $g(x) = 3 + 2x + x^2 + 8x^3 = 0$, so the sum $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{-1}{8}$.

18. Two students are playing a game. They take a deck of five cards numbered 1 through 5, shuffle them, and then place them in a stack facedown, turning over the top card next to the stack. They then take turns either drawing the card at the top of the stack into their hand, showing the drawn card to the other player, or drawing the card that is faceup, replacing it with the card on the top of the pile. This is repeated until all cards are drawn, and the player with the largest sum for their cards wins. What is the probability that the player who goes second wins, assuming optimal play?

Answer: $\frac{4}{15}$

Solution:

The only possible losing hands for the first player are $\{1, 2, 3\}$ and $\{1, 2, 4\}$, so the first player will draw from whatever pile minimizes their chance of ending up with these cards. We consider this by cases. Suppose first that the 5 is revealed initially. Then the first player draws the 5 and cannot get a losing hand, so there is a 0 probability that the second player wins.

If either a 1 or a 2 is revealed, the first player will draw from the facedown pile, since drawing revealed card increases their chances of having a losing hand. The other player will also draw from the facedown pile, since they must force the first player to draw the revealed card. Thus the game comes down entirely to the order of the cards in the facedown pile. Assume WLOG the revealed card was a 1. Then for the first player to draw the 2, the 2 card must appear in either the first or third position in the facedown pile, which has probability $1/2$. Then either the 4 or the 3 must appear in the other odd position, which has probability of $2/3$. Thus the probability that the first player loses in these cases is $1/3$.

If either the 3 or the 4 is revealed, the first player will draw the revealed card, since it decreases the number of possible losing hands. Assume WLOG the revealed card was the 3. Then for the second player to win, they must draw the 4 and the 5. If the 4 and the 5 are in the first and second position in the pile of four cards remaining, then the second player loses, since they will draw the first card and player one will draw the second. If the 4 and the 5 are in the first and third position, then they will also lose, since they will draw the first card, when it is revealed, the second card will be revealed to be a 1 or a 2, and the first player will draw the either the 4 or the 5 from the top of the deck. If the 4 and the 5 are in the first and the fourth position, though, then the second player will win. Likewise, the second player loses if the 4 and the 5 are in the second and third positions or in the third and fourth positions, but wins if the 4 and the 5 are in second and fourth positions. Thus the probability that the second player wins in these cases is $1/3$.

Averaging these probabilities, we get that the probability that the second player wins is $(4(1/3) + 0)/5 = 4/15$.

19. Compute the sum of all primes p such that $2^p + p^2$ is also prime.

    Answer: 3

    Solution: Note if $p > 3$, then $p \equiv 1$ or $5 \mod 6$. Then $p^2 \equiv 1 \mod 3$. As p is odd, $2^p \equiv 2 \mod 3$, so $p^2 + 2^p$ is divisible by 3 and thus not prime. Thus we only need to check the cases when $p = 2$ and $p = 3$, and we see that $p^2 + 2^p$ is only prime when $p = \boxed{3}$ $(p^2 + 2^p = 17)$.

20. In how many ways can one color the 8 vertices of an octagon each red, black, and white, such that no two adjacent sides are the same color?

    Answer: 258

    Solution: We show this through a recursive equation.

    For a '2-gon', there are 3 ways to colour the first vertice and 2 ways to colour the second vertice, so there are 6 ways.

For a triangle, there are 3 ways to colour the first vertice, 2 ways to colour the second vertice, and the third vertice is forced, so there are 6 ways.

For any n-polygon with four or more sides, we consider two cases.
Case 1: Vertex 1 and vertex $n-1$ are different colours. Then this means that vertices 1 to $n-1$ can be coloured in all ways possible for a $n-1$ gon and the colour of vertex n is fixed.
Case 2: Vertex 1 and vertex $n-1$ are the same colour. Then vertices 1 to $n-2$ can be coloured in all ways possible for a $n-2$ gon, and then there are 2 ways to colour vertex n.

We define a_n as the legal colourings of an n-gon. Then a_n can be recursively defined as

$$a_n = a_{n-1} + 2a_{n-2}$$

Therefore, for $n = 8$, $a_n = 258$.

21. If $f = \cos(\sin(x))$. Calculate the sum

$$\sum_{n=0}^{2021} f''(n\pi).$$

Answer: -2022

Solution: We can see from the chain rule that there will always be a sin factor

$$f'(x) = -\sin(\sin(x)) \cdot \cos(x)$$

$$f''(x) = -\cos(\sin(x)) \cdot \cos^2(x) + \sin(\sin(x)) \cdot \sin(x)$$

So, we can see that at $n\pi$, $\sin(n\pi) = 0$. So the second factor is always 0 and the first factor is $-\cos(0) \cdot \cos^2(n\pi) = -1$. So, the sum is simply $\sum_{n=0}^{2021} -1 = \boxed{-2022}$.

22. Find all real values of A that minimize the difference between the local maximum and local minimum of $f(x) = (3x^2 - 4)(x - A + \frac{1}{A})$.

Answer: $A = \pm 1$

Solution 1: This question has a few different solutions. The first three will focus on the substitution $A - \frac{1}{A} = k$, while the last one will be a "bashy" solution. Note that Solution 1 and Solution 2 are *not* rigorous solutions with reasonable intuition. They are ways to quickly solve the problem if the solver makes huge assumptions. I include these solutions because the assumptions make the problem "cleaner," so a potential solver might use them if they can't otherwise solve the problem.

This first solution is not rigorous, but I guess it might work as a "trick" if there's a method I don't know about that lets you make the following assumption.

Replace $A - \frac{1}{A}$ with k. If we can solve the question for k, we can solve it for A as well. Assume the local maximum and minimum are closest together when $k = 0$ ($f(x) = (3x^2 - 4)(x)$). Then, $\frac{1}{A} - A = 0$. We can then observe or calculate using a quadratic that $\boxed{A = \pm 1}$. This seems stupid, but I thought I would include it anyway because assuming $k = 0$ does look very neat if you don't know how to solve it.

Solution 2: This solution is similarly not rigorous but a possible way to "solve" it. One might notice that we know where all of the roots of f are: $\frac{2\sqrt{3}}{3}$, $\frac{-2\sqrt{3}}{3}$, and $-A+\frac{1}{A}$. These are equally spaced along the x axis when $-A+\frac{1}{A}=0$. If you assume this is where the minimum difference between the local maximum and minimum of f is, you can then solve for A to get $\boxed{A=\pm 1}$. Again, this seems stupid but I thought it might be worth including because it's something I thought about afterwards.

Solution 3: These next two solutions are "true" solutions in the sense that they are rigorous. Begin by making the substitution with k as described in Solution 1. Then,

$$f(x)=(3x^2-4)(x-k).$$

To find where the local minimum and local maximum are, take the derivative of f relative to x and set it equal to zero.

Using the product rule and then simple algebra,

$$f'(x)=(6x)(x-k)+(3x^2-4)=0$$

$$f'(x)=9x^2-6kx-4=0.$$

$$x=\frac{6k\pm\sqrt{36k^2+144}}{18}$$

$$x=\frac{k\pm\sqrt{k^2+4}}{3}$$

This will be used later.

Then, supposed x_1 and x_2 are the two roots of $f'(x)$. Define $x_1<x_2$. We know that because f is a cubic with a leading coefficient of 1, so $f(x_1)>f(x_2)$. Therefore, the difference between the local maximum and local minimum of f is

$$f(x_1)-f(x_2).$$

Performing simple algebra,

$$(3x_1^2-4)(x_1-k)-(3x_2^2-4)(x_2-k)$$

$$(3x_1^3-4x_1-3x_2^3+4x_2)-k(3x_1^2-4-3x_2^2+4)$$

$$(3x_1^3-4x_1-3x_2^3+4x_2)-3k(x_1^2-x_2^2)$$

$$(3x_1^3-4x_1-3x_2^3+4x_2)-3k(x_1+x_2)(x_1-x_2)$$

Because $x_1<x_2$, we know that $x_1-x_2=\frac{-2\sqrt{k^2+4}}{3}$. Therefore, substitute x_1-x_2 with this.

$$(3x_1^3-4x_2-3x_2^3+4x_2)-k(x_1+x_2)(-2\sqrt{k^2+4})$$

From the derivative we took of f, we can use Vieta's relationships to find:

$$\frac{6k}{9}=\frac{2k}{3}=x_1+x_2$$

Substitute this into the equation for $f(x_1)-f(x_2)$.

$$(3x_1^3 - 4x_1 - 3x_2^3 + 4x_2) + \frac{4k^2\sqrt{k^2+4}}{3}$$
$$(3(x_1^3 - x_2^3) - 4(x_1 - x_2) + \frac{4k^2\sqrt{k^2+4}}{3}$$
$$(x_1 - x_2)(3(x_1^2 + x_1x_2 + x_2^2) - 4) + \frac{4k^2\sqrt{k^2+4}}{3}$$
$$(x_1 - x_2)(3((x_1 + x_2)^2 - x_1x_2) - 4) + \frac{4k^2\sqrt{k^2+4}}{3}$$

We know $x_1x_2 = \frac{-4}{9}$ and $x_1 + x_2 = \frac{2k}{3}$ from Vieta's relationships.

$$(x_1 - x_2)(3((\frac{2k}{3})^2 - \frac{-4}{9}) - 4) + \frac{4k^2\sqrt{k^2+4}}{3}$$
$$(x_1 - x_2)(\frac{4k^2 - 8}{3}) + \frac{4k^2\sqrt{k^2+4}}{3}$$

We can also calculate $x_1 - x_2$ because we know both x_1 and x_2 from the quadratic equation earlier. This value is $\frac{-2\sqrt{k^2+4}}{3}$.

$$(\frac{-2\sqrt{k^2+4}}{3})(\frac{4k^2 - 8}{3}) + \frac{4k^2\sqrt{k^2+4}}{3}$$
$$\frac{-8k^2\sqrt{k^2+4} + 16\sqrt{k^2+4}}{9} + \frac{12k^2\sqrt{k^2+4}}{9}$$
$$\frac{4k^2\sqrt{k^2+4} + 16\sqrt{k^2+4}}{9}$$
$$\frac{4}{9}(\sqrt{k^2+4})(k^2+4)$$

At this point one can either differentiate to show that it is minimized when $k = 0$ or observe that because $k^2 \geq 0\,\forall k$, the value of k that minimizes this is $k^2 = 0 \Rightarrow k = 0$.

From this point, one can either solve for A knowing that $A - \frac{1}{A} = 0$ or observe that $\boxed{A = \pm 1}$.

Solution 4: This method is based around not making the substitution $k = A - \frac{1}{A}$. This was how I initially solved the problem and, I guess, the "intended" solution.

First, find $f'(x)$ and set it equal to 0.

$$f'(x) = (6x)(x - A + \frac{1}{A}) + (3x^2 - 4)$$
$$6x^2 - 6A + \frac{6}{A} + 3x^2 - 4 = 0$$
$$9x^2 - 6(A - \frac{1}{A}) - 4 = 0$$

Using the quadratic formula,

$$x = \frac{6A - \frac{6}{A} \pm \sqrt{36(A^2 + \frac{1}{A^2} + 2)}}{18}$$
$$x = \frac{6A - \frac{6}{A} \pm 6(A + \frac{1}{A})}{18}$$
$$x = \frac{A - \frac{1}{A} \pm (A + \frac{1}{A})}{3}$$
$$x = \frac{2A}{3}, \frac{-2}{3A}$$

Now, we can plug these values of A back into f and subtract. We don't know which one is positive, because we don't know if $A > 0$, so we don't know whether to subtract $f(\frac{2A}{3})$ from $f(\frac{-2}{3A})$ or vice-versa. However, we know the difference must be positive, so we can take the absolute value of their difference.

Minimize:

$$|f(\frac{2A}{3}) - f(\frac{-2}{3A})|$$
$$|(3(\frac{2A}{3})^2 - 4)(\frac{2A}{3} - A + \frac{1}{A}) - (3(\frac{-2}{3A})^2 - 4)(\frac{-2}{3A} - A + \frac{1}{A})|$$
$$|(\frac{4A^2}{3} - 4)(\frac{1}{A} - \frac{A}{3}) - (\frac{4}{3A^2} - 4)(\frac{1}{3A} - A)|$$

The content of this absolute value can be rewritten using AM-GM. Treat the first term of AM-GM as the left term and the second term of AM-GM as the second term (including the negative). Then, it is known that:

$$\frac{(\frac{4A^2}{3} - 4)(\frac{1}{A} - \frac{A}{3}) - (\frac{4}{3A^2} - 4)(\frac{1}{3A} - A)}{2} \geq \sqrt{(-1)(\frac{4A^2}{3} - 4)(\frac{1}{A} - \frac{A}{3})(\frac{4}{3A^2} - 4)(\frac{1}{3A} - A)}$$

For the next algebra, I will only operate on the right hand side.

$$4\sqrt{(-1)(\frac{A^2}{3} - 1)(\frac{1}{A} - \frac{A}{3})(\frac{1}{3A^2} - 1)(\frac{1}{3A} - A)}$$

Rearranging the terms for more simple multiplication,

$$4\sqrt{(-1)(\frac{A^2}{3} - 1)(\frac{1}{3A^2} - 1)(\frac{1}{3A} - A)(\frac{1}{A} - \frac{A}{3})}$$

Then, multiply the first two parenthesis groups together and the last two parenthesis parenthesis groups together.

$$4\sqrt{(-1)(\frac{1}{9} - \frac{A^2}{3} - \frac{1}{3A^2} + 1)(\frac{1}{3A^2} - \frac{1}{9} - 1 + \frac{A^2}{3})}$$
$$4\sqrt{(\frac{A^2}{3} + \frac{1}{3A^2} - \frac{10}{9})^2}$$
$$4(\frac{A^2}{3} + \frac{1}{3A^2} - \frac{10}{9})$$

Putting this back into the full AM-GM,

$$\frac{(\frac{4A^2}{3}-4)(\frac{1}{A}-\frac{A}{3})-(\frac{4}{3A^2}-4)(\frac{1}{3A}-A)}{2} \geq 4(\frac{A^2}{3}+\frac{1}{3A^2}-\frac{10}{9})$$

$$(\frac{4A^2}{3}-4)(\frac{1}{A}-\frac{A}{3})-(\frac{4}{3A^2}-4)(\frac{1}{3A}-A) \geq 8(\frac{A^2}{3}+\frac{1}{3A^2}-\frac{10}{9})$$

Because we want to minimize the left hand side (remember, the absolute value of that is the difference between the the local maximum and minimum of f), we want the right hand side to be as small as possible to show the minimum value the left hand side could possibly take.

Minimize:

$$8(\frac{A^2}{3}+\frac{1}{3A^2}-\frac{10}{9})$$

This is clearly minimized when $\frac{A^2}{3}+\frac{1}{3A^2}$ is minimized. It can be seen either through applying AM-GM to this pair in a similar manner or through observation that $\frac{A^2}{3}+\frac{1}{3A^2}$ is minimized when $A = \pm 1$. To be thoroughly rigorous, though, let's do the AM-GM.

$$\frac{\frac{A^2}{3}+\frac{1}{3A^2}}{2} \geq \sqrt{(\frac{A^2}{3})(\frac{1}{3A^2})}$$

$$\frac{\frac{A^2}{3}+\frac{1}{3A^2}}{2} \geq \frac{1}{3}$$

$$A^2+\frac{1}{A^2} \geq 2$$

Substituting $M = A^2$,

$$M+\frac{1}{M} \geq 2$$

We want this to equal 2 to minimize $A^2 = M$.

$$M+\frac{1}{M} = 2$$

$$M^2-2M+1=0$$

$$(M-1)^2=0$$

$$M = A^2 = 1 \Rightarrow A = \pm 1$$

Therefore, for logic stated earlier, $\boxed{A = \pm 1}$ minimizes the expression on the left of the first AM-GM, which minimizes the difference between the local maximum and minimum of f.

23. Bessie is playing a game. She labels a square with vertices labeled A, B, C, D in clockwise order. There are 7 possible moves: she can rotate her square 90 degrees about the center, 180 degrees about the center, 270 degrees about the center; or she can flip across diagonal AC, flip across diagonal BD, flip the square horizontally (flip the square so that vertices A and B are switched and vertices C and D are switched), or flip the square vertically (vertices B and C are switched, vertices A and D are switched). In how many ways can Bessie arrive back at the original square for the first time in 3 moves?

Answer: 42

Solution: If Bessie makes two moves, she will arrive at a square that is another move of the game or the original square. We can disregard the case where we arrive back at the original square (because there is no way to move to the original square in one more move and we are looking for the ways we can arrive back at the original square for the first time in three moves). If we arrive at a square that is another move after two moves, there is only one possible move to make to arrive back at the original square. Thus, we can count the number of ways we can make two moves such that we do not arrive back at the original square.

In total, there are $(7 * 7 = 49)$ combinations of two moves we can make. Of these, 7 result in arriving back at the original square. These are rotate 90, rotate 270; rotate 180, rotate 180; rotate 270, rotate 90; flip across AC, flip across AC; flip across BD, flip across BD; flip horizontally, flip horizontally; flip vertically, flip vertically.

Thus we conclude there are $49 - 7 = 42$ ways to arrive back at the original square for the first time in 3 moves.

For further reference, we can consult the Cayley Table for D4 [insert potential image]. Another note is that if we arrive at the original square in three moves, we will always be arriving at the original square for the first time.

24. A positive integer is called *happy* if the sum of its digits equals the two-digit integer formed by its two leftmost digits. Find the number of 5-digit happy integers.

Answer: 1110

Solution: Denote a five-digit positive integer by $\overline{abcde}$, where a, b, c, d, and e are its digits. Then the condition given in the problem is equivalent to the equation

$$10a + b = a + b + c + d + e,$$

since $10a + b$ represents concatenating the first two digits, and $a + b + c + d + e$ is the digital sum of the entire integer. This leads to, after combining like terms,

$$9a = c + d + e,$$

so $c + d + e$ must be positive and divisible by 9. We have three cases:

Case 1: $c + d + e = 9$. By stars-and-bars, there are $\binom{9+3-1}{3-1} = \binom{11}{2} = 55$ solutions to this equation in nonnegative integers.

Case 2: $c + d + e = 18$. This is a little more complicated, since $c, d, e \leq 9$, so we first compute the number of distributions without restriction and then subtract the solutions in which at least one of c, d, or e is greater than 9. There are $\binom{20}{2} = 190$ solutions. Now, note that exactly one of c, d, or e can be greater than 9. There are 3 ways to choose which one and $\binom{10}{2} = 45$ to complete the rest of the distribution among the three integers, so we subtract $3 \times 45 = 135$ from 190, giving us 55 again.

Case 3: $c + d + e = 27$. There is only one way to achieve this, namely $c = d = e = 9$.

Hence, there are $55 + 55 + 1 = 111$ ways the sum of the three rightmost digits can be a multiple of 9, after which a is fixed and b can be any digit from 0 to 9, inclusive, so the total number of 5-digit happy integers is $111 \times 10 = \boxed{1110}$.

25. Compute:

$$\frac{\sum_{i=0}^{\infty} \frac{(2\pi)^{4i+1}}{(4i+1)!}}{\sum_{i=0}^{\infty} \frac{(2\pi)^{4i+1}}{(4i+3)!}}$$

Answer: $4\pi^2$

Solution:

The Taylor series for $\sin(x)$ is

$$\sin(x) = \sum_{i=0}^{\infty} \frac{x^{2i+1}(-1)^i}{(2i+1)!}$$

Substituting 2π for x, we see:

$$\sin(2\pi) = \sum_{i=0}^{\infty} \frac{(2\pi)^{4i+1}}{(4i+1)!} - \sum_{i=0}^{\infty} \frac{(2\pi)^{4i+3}}{(4i+3)!} = 0$$

And then rearranging terms,

$$\sum_{i=0}^{\infty} \frac{(2\pi)^{4i+1}}{(4i+1)!} = (2\pi)^2 \sum_{i=0}^{\infty} \frac{(2\pi)^{4i+1}}{(4i+3)!}$$

$$\frac{\sum_{i=0}^{\infty} \frac{(2\pi)^{4i+1}}{(4i+1)!}}{\sum_{i=0}^{\infty} \frac{(2\pi)^{4i+1}}{(4i+3)!}} = (2\pi)^2$$

is our answer.

26. Suppose points A, B, C, D lie on a circle ω with radius 4 such that $ABCD$ is a quadrilateral with $AB = 6, AC = 8, AD = 7$. Let E and F be points on ω such that AE and AF are respectively the angle bisectors of $\angle BAC$ and $\angle DAC$. Compute the area of quadrilateral $AECF$.

Answer: $4\sqrt{7} + 2\sqrt{15}$

Solution: Since AC is a diameter of ω, it follows that $\triangle ABC, \triangle AEC, \triangle AFC, \triangle ADC$ are all right triangles with AC as the hypotenuse. Letting $\alpha = \angle EAC$ and $\beta = \angle FAC$, it follows that the area of quadrilateral $AECF$ is equal to

$$\begin{aligned}\frac{(AE)(EC) + (AF)(FC)}{2} &= \frac{(ACsin\alpha)(ACcos\alpha) + (ACsin\beta)(ACcos\beta)}{2} \\ &= \frac{AC^2}{4}(sin(2\alpha) + sin(2\beta)) \\ &= \frac{AC^2}{4}\left(\frac{BC}{AC} + \frac{DC}{AC}\right) \\ &= \frac{AC(BC + DC)}{4}.\end{aligned}$$

Using Pythagorean Theorem, it follows that $BC = \sqrt{8^2 - 6^2} = 2\sqrt{7}$ and $DC = \sqrt{8^2 - 7^2} = \sqrt{15}$. Plugging in these values, the area of quadrilateral $AECF$ is equal to $\frac{8(2\sqrt{7}+\sqrt{15})}{4} = 4\sqrt{7}+2\sqrt{15}$.

27. Let $P(x) = x^2 - ax + 8$ with a a positive integer, and suppose that P has two distinct real roots r and s. Points $(r, 0)$, $(0, s)$, and (t, t) for some positive integer t are selected on the coordinate plane to form a triangle with an area of 2021. Determine the minimum possible value of $a + t$.

Answer: 129

Solution: For simplicity, let $A = (r, 0)$, $B = (0, s)$, and $C = (t, t)$. By a standard distance formula, the altitude from C to $\overline{AB}$ has length $\frac{|st + rt - rs|}{\sqrt{r^2 + s^2}}$. Substituting $8 = rs$ and $a = r + s$ gives the altitude's length as $\frac{|at - 8|}{\sqrt{r^2 + s^2}}$. Now, since $AB = \sqrt{r^2 + s^2}$, the area of triangle $\triangle ABC$ is equal to $\frac{|at - 8|}{2}$. We are given that this area is 2021, so $|at - 8| = 4042$, and $at = 4050$ or -4036. Since both a and t are positive integers, we can ignore the latter case and conclude that $at = 4050$. The minimum value of $a + t$ is thus obtained when the positive difference between them is as small as possible, which occurs when $a = 75$ and $b = 54$ (or vice versa). Hence, $a + t \geq 75 + 54 = 129$.

28. A quartic $p(x)$ has a double root at $x = -\frac{21}{4}$, and $p(x) - 1344x$ has two double roots each $\frac{1}{4}$ less than an integer. What are these two double roots?

Answer: $\mathbf{\frac{3}{4}, -\frac{77}{4}}$**.**

Solution: Let $q(x) = p(x - \frac{21}{4})$ so $q(x)$ has a double root at 0, i.e. can be written as $x^2 r(x)$ for some quadratic $r(x)$. The second condition says that $q(x + \frac{21}{4}) - 1344x$ has two double roots each $\frac{1}{4}$ less than an integer, i.e. $q(x) - 1344x + 7056$ has two integer double roots so can be written as $(x - c)^2(x - d)^2$ for integers c, d. Notice that when we expand $(x - c)^2(x - d)^2$, the degree 2 through 4 terms must be part of $q(x) = x^2 r(x)$ and the degree 0 and 1 terms match with $-1344x + 7056$. More explicitly, we have that

$$-1344 = -(2c^2 d + 2cd^2) \qquad \text{and} \qquad 7056 = c^2 d^2.$$

We see that $cd = \pm 84$ and $c + d = \pm 8$ is only solved with $c = 6, d = -14$ (or $d = 6, c = -14$). Thus, the desired double roots are $6 - \frac{21}{4}$ and $-14 - \frac{21}{4}$, i.e. $\boxed{\frac{3}{4}, -\frac{77}{4}}$.

29. Consider pentagon ABCDE. How many paths are there from vertex A to vertex E where no edge is repeated and does not go through E.

Answer: 58

Solution: From the starting vertex A, the next vertex of the path has two options: 1) E or 2) B, C, or D. In the first case there is one such path.

In the second, assume without loss of generality that the next vertex of the path is B. Then, either the next vertex is 21) E or 22) C or D. The first case has one such path.

Case 22 assume without loss of generality that the next vertex is C. Then, the next vertex is either 221) E, 222) D, or 223) A. There is only one such path is case 221.

In case 222 the next vertex can be either 2221) B or 2222) A. In the first case, there is only one such path. In the second case, the remainder of the path can either be E or C then E, for two total possible paths.

In case 223 the next vertex can be either 22231) E or 2232) D. The first case has only one such path. If the next vertex is D then the remainder of the path can either be BE or CE, for two possible paths.

Therefore, the total number of desired paths is $1+3*(1+2*(1+(1+1+1+1)+(1+1+2*1))) = \boxed{58}$

30. Let $a_1, a_2, \ldots$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n = 4$. Compute the maximum possible value of $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{2^n}$ (assume this always converges).

Answer: $\frac{2\sqrt{3}}{3}$

Solution: By Cauchy-Schwarz,

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{2^n} \leq \Big(\sum_{n=1}^{\infty} a_n\Big)^{\frac{1}{2}} \Big(\sum_{n=1}^{\infty} \frac{1}{2^{2n}}\Big)^{\frac{1}{2}}.$$

Then the right hand side equals

$$\left(4^{\frac{1}{2}}\right)\left(\frac{1}{3}\right)^{\frac{1}{2}} = \boxed{\frac{2\sqrt{3}}{3}}.$$

This value is obtained by the series $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{2^n}$ when $a_n = \frac{12}{2^{2n}}$.

31. Define function $f(x) = x^4 + 4$. Let

$$P = \prod_{k=1}^{2021} \frac{f(4k-1)}{f(4k-3)}.$$

Find the remainder when P is divided by 1000.

Answer: Solution found: 57

Solution:

32. Reduce the following expression to a simplified rational:

$$\cos^7 \frac{\pi}{9} + \cos^7 \frac{5\pi}{9} + \cos^7 \frac{7\pi}{9}$$

Answer: $\boxed{\frac{63}{128}}$

Solution: We observe that the angles are all 1/3 of a nice angles – in fact, $\cos 3\theta_i = \frac{1}{2}$ for all i. Letting $x_i = \cos\theta_i$, noting that x_i are all distinct, and using the triple angle formula gives us the following polynomial relationship:

$$4x_i^3 - 3x_i = \frac{1}{2}.$$

Define $S_n = \sum_i x_i^n$. We wish to compute S_7. The polynomial relationship above gives us a nice recursion for S_n:

$$x_i^3 = \frac{3}{4}x_i + \frac{1}{8} \Rightarrow x_i^n = \frac{3}{4}x_i^{n-2} + \frac{1}{8}x_i^{n-3} \Rightarrow S_n = \frac{3}{4}S_{n-2} + \frac{1}{8}S_{n-3}.$$

We have $S_0 = 3$ trivially, and $S_1 = 0$ follows from Viete's formula applied to $x^3 - \frac{3}{4}x - \frac{1}{8}$. We can find S_2 using Viete's formula as well: $x_1x_2 + x_2x_3 + x_3x_1 = -\frac{3}{4}$. Then $S_2 = S_1^2 - 2 \cdot (-\frac{3}{4}) = \frac{3}{2}$. With the first three values known, we can apply the recursion to find the rest!

$$S_3 = \frac{3}{4}S_1 - \frac{1}{8}S_0 = \frac{3}{8},$$
$$S_4 = \frac{9}{8},$$
$$S_5 = \frac{15}{32},$$
$$S_6 = \frac{57}{64},$$
$$S_7 = \boxed{\frac{63}{128}}.$$

33. Lines ℓ_1 and ℓ_2 have slopes m_1 and m_2 such that $0 < m_2 < m_1$. ℓ_1' and ℓ_2' are the reflections of ℓ_1 and ℓ_2 about the line ℓ_3 defined by $y = x$. Let $A = \ell_1 \cap \ell_2 = (5, 4)$, $B = \ell_1 \cap \ell_3$, $C = \ell_1' \cap \ell_2'$ and $D = \ell_2 \cap \ell_3$. If $\frac{4-5m_1}{-5-4m_1} = m_2$ and $\frac{(1+m_1^2)(1+m_2^2)}{(1-m_1)^2(1-m_2)^2} = 41$, compute the area of quadrilateral $ABCD$.

Answer: 4

Solution: We rearrange the first equation to get $\frac{m_2-m_1}{1+m_1m_2} = -\frac{4}{5}$. If we set $m_1 = \tan\alpha$ and $m_2 = \tan\beta$ where $\alpha > \beta$ and α, β are the acute angles ℓ_1 and ℓ_2 make with the x-axis, then this is equivalent to $\tan(\beta - \alpha) = \tan(180 + (\beta - \alpha)) = \tan(\beta + (180 - \alpha)) = \tan\angle DAB$. Thus $\sin\angle DAB = \frac{4}{\sqrt{41}}$. We now compute the lengths of AB and AD usiing the distance formula. If $\ell_1 := m_1x + b_1$ and $\ell_2 := m_2x + b_2$, we can compute $B = \left(\frac{b_1}{1-m_1}, \frac{b_1}{1-m_1}\right)$ and $D = \left(\frac{b_2}{1-m_2}, \frac{b_2}{1-m_2}\right)$. Since $\ell_1 \cap \ell_2 = (5, 4)$ we have $b_1 = 4 - 5m_1$ and $b_2 = 4 - 5m_2$. Plugging into the distance formula and doing some algebra gives $AB = \frac{\sqrt{1+m_1^2}}{1-m_1}$ and $AD = \frac{\sqrt{1+m_2^2}}{1-m_2}$. Now by the sine area formula, $[ABCD] = [DAB] + [BCD] = AB \cdot CD \cdot \sin\angle DAB = \sqrt{41} \cdot \frac{4}{\sqrt{41}} = \boxed{4}$.

34. Suppose $S(m, n) = \sum_{i=1}^{m} (-1)^i i^n$. Compute the remainder when $S(2020, 4)$ is divided by $S(1010, 2)$.

Answer: 509545

Solution: First, $S(1010, 2) = 1010^2 - 1009^2 + \ldots + 2^2 - 1^2$, which by difference of squares is

$$(1010 + 1009) + \ldots + (2 + 1) = \sum_{i=1}^{1010} i = \frac{1}{2}(1010)(1011) = 505 \cdot 1011.$$

Furthermore, $S(2020, 4) = S(2020, 0) \cdot (2S(2020, 2) - 1)$, where

$$S(2020, 2) = \frac{1}{2}(2020)(2021) = 1010 \cdot 2021.$$

Since $S(2020, 2)$ is a multiple of 505, it is congruent to 0 mod 505. Furthermore, $S(2020, 2) \equiv -1 \cdot -1 \equiv 1 \bmod 1011$. Since $505 \cdot 2 \equiv -1 \bmod 1011$, it follows by CRT that $-1010 \equiv S(1010, 2) - 1010$ is the remainder, which is the value 509545.

35. Let N be the number of ways to place the numbers $1, 2, ..., 12$ on a circle such that every pair of adjacent numbers has greatest common divisor 1. What is N/144? (Arrangements that can be rotated to yield each other are the same).

Answer: 240

Solution: First, begin by noticing that no pair of even numbers can be adjacent. Each even number is adjacent to odd numbers; each odd number is adjacent to even numbers. (In other words, the ordering of numbers on the circle is even number, odd number, even number, odd number, etc...) Of the even numbers, 6, 12, and 10 are particularly interesting. This is because if we know where 6, 10, and 12 are positioned, we can eliminate possibilities for where 3 and 5 are positioned. Now we begin casework.

For sake of clarity, label the 12 positions on the circle as $1, ..., 12$. For each case, we will first place 6, 12, and 10 at certain positions. Then we will count possibilities for 3 and 5. Finally we will count the number of ways to change 6, 12 and 10.

Case 1: Let 6 be in position 1 and 12 be in position 3. Let 10 be in position 5. In this type of arrangement, there are 2 ways to place 3 and 9 and 2 ways to place 5, or there are 4 ways to place 3 and 9 and 3 ways to place 5. The other 4 odd numbers can be placed in any of the remaining even positions and the other 3 even numbers can be placed in any of the remaining odd positions. Finally, there are 4 ways to change 12, 6, and 10. In total, this yields $(4 + 12) * 4 * 4! * 3!$ ways.

Case 2: 6 in position 1, 12 in position 3, 10 in position 7. A similar analysis should yield $(12 + 8) * 4 * 4! * 3!$ ways.

Case 3: 6 in position 1, 12 in position 5, 10 in position 7. A similar analysis should yield $6 * 4 * 4! * 3!$ ways.

Case 4: 6 in position 1, 12 in position 5, 10 in position 9. A similar analysis should yield $8 * 2 * 4! * 3!$ ways.

Case 5: 6 in position 1, 10 in position 3, 12 in position 5. A similar analysis should yield $4 * 2 * 4! * 3!$ ways.

Case 6: 6 in position 1, 12 in position 7, 10 in position 9. A similar analysis should yield $6 * 8 * 4! * 3!$ ways.

Summing up all the cases we get $[(4+12)*4+(12+8)*4+6*4+8*2+4*2+6*8]*4!*3! = 240*144$. So $N/144 = 240$.

36. Compute the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\binom{2n}{2}} = \frac{1}{\binom{2}{2}} - \frac{1}{\binom{4}{2}} + \frac{1}{\binom{6}{2}} - \frac{1}{\binom{8}{2}} - \frac{1}{\binom{10}{2}} + \frac{1}{\binom{12}{2}} + \cdots .$$

Answer: $\boxed{\frac{\pi}{2} - \ln 2}$

Solution: Recall the Maclaurin series expansion for $\tan^{-1} x$:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

Integrating the right-hand side yields:

$$\int_0^1 \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) dx = \left[\frac{x^2}{2\cdot 1} - \frac{x^4}{4\cdot 3} + \frac{x^6}{6\cdot 5} - \frac{x^8}{8\cdot 7} + \cdots\right]_0^1$$
$$= \frac{1}{2}\left(\frac{1}{\binom{2}{2}} - \frac{1}{\binom{4}{2}} + \frac{1}{\binom{6}{2}} - \frac{1}{\binom{8}{2}} + \cdots\right).$$

Hence, the series we wish to calculate is given by twice the integral of $\tan^{-1} x$ from 0 to 1. This can be found by integration by parts. We let $u = \tan^{-1}(x)dx$ and $dv = 1$ to get

$$\begin{aligned}\int_0^1 \tan^{-1} x &= \int_0^1 udv \\ &= uv\Big|_0^1 - \int_0^1 vdu \\ &= x\tan^{-1} x\Big|_0^1 - \int_0^1 \frac{xdx}{1+x^2} \\ &= \left[x\tan^{-1} x - \frac{1}{2}\ln(1+x^2)\right]_0^1 \\ &= \frac{\pi}{4} - \frac{\ln 2}{2}.\end{aligned}$$

Thus, the series is equal to twice this quantity, or

$$\frac{\pi}{2} - \ln 2.$$