1. A paper rectangle $ABCD$ has $AB = 8$ and $BC = 6$. After corner B is folded over diagonal AC, what is BD?

   Answer: $\mathbf{\frac{14}{5}}$

   Solution: Use the law of cosines on $\triangle BCD$, so

   $$BD = \sqrt{BC^2 + DC^2 - 2(BC)(DC)\cos(90^\circ - 2x)}$$

   where $x = \angle DCA$ and $\angle BCA = 90^\circ - x$. Plugging in $\cos(90^\circ - 2x) = \sin(2x) = \frac{24}{25}$, $BD = \boxed{\frac{14}{5}}$.

2. Let $ABCD$ be a trapezoid with bases $AB = 50$ and $CD = 125$, and legs $AD = 45$ and $BC = 60$. Find the area of the intersection between the circle centered at B with radius BD and the circle centered at D with radius BD. Express your answer as a common fraction in simplest radical form and in terms of π.

   Answer: $\mathbf{\frac{14{,}450\pi}{3} - \frac{7{,}225\sqrt{3}}{2}}$

   Solution: Drop the altitude from A to CD and call that point E. Drop the altitude from B to CD and call that point F.

   Thus, $ABFE$ is a rectangle and $EF = 50$. This implies that $DE+FC = CD-EF = 125-50 = 75$.

   Now, combine the two triangles AED and BFC along AE and BF which is equal to the height of the trapezoid. This triangle has side lengths 45, 60 and 75 which form a pythagorean triple that is a multiple of $(3-4-5)$.

   The height of the trapezoid is the height of the combined triangle to the side with length 75. $BF * 75 = AE * 75 = 45 * 60$. $AE = BF = \frac{45*60}{75} = 36$.

   Notice that triangles AED and CFB are similar to the combined triangle. This implies that AED and CFB have side lengths that are pythagorean triples that are multiples of $(3-4-5)$.

   Using this fact, it can be found that $DE = 27$ and $CF = 48$ where $DE + CF = 27 + 48 = 75$, as we found above.

   Next, BD can be found from applying the pythagorean theorem to triangle BFD. $BF = 36$ and $DF = DE + EF = 27 + 50 = 77$.

   Using the pythagorean theorem: $BD = \sqrt{36^2 + 77^2} = \sqrt{1,296 + 5,929} = \sqrt{7,225} = 85$.

   Now that the radius has been found, the area of the intersection can be found.

   The intersection will compose of two equilateral triangles with side length 85 and four pieces calculated from subtracting an equilateral triangle with side length 85 from $\frac{1}{6}$ of the area of the circle. Call the area of an equilateral triangle A and the area of a piece P.

   The answer is thus, total area calculated from $2A + 4P$.

   $A = \frac{85^2\sqrt{3}}{4} = \frac{7{,}225\sqrt{3}}{4}$.

   $P = \frac{1}{6}(85^2\pi) - \frac{85^2\sqrt{3}}{4} = \frac{7{,}225\pi}{6} - \frac{7{,}225\sqrt{3}}{4}$

   Thus, $2A + 4P = 2(\frac{7{,}225\sqrt{3}}{4}) + 4(\frac{7{,}225\pi}{6} - \frac{7{,}225\sqrt{3}}{4}) = \frac{14{,}450\pi}{3} - \frac{7{,}225\sqrt{3}}{2}$

3. If r is a rational number, let $f(r) = (\frac{1-r^2}{1+r^2}, \frac{2r}{1+r^2})$. Then the images of f forms a curve in the xy plane. If $f(1/3) = p_1$ and $f(2) = p_2$, what is the distance along the curve between p_1 and p_2?

Answer: $\pi/2$

Solution: First we note that $\frac{1-r^2}{1+r^2}^2 + \frac{2r}{1+r^2}^2 = \frac{1-2r^2+r^4+4r^2}{1+r^2}^2 = 1$. So, the curve is the unit circle. Then $1/3$ maps to $(8/10, 6/10)$ and 2 maps to $(-3/5, 4/5)$. Thus, this is $1/4$ of the circumference and the distance is $\boxed{\pi/2}$.

4. $\triangle A_0B_0C_0$ has side lengths $A_0B_0 = 13$, $B_0C_0 = 14$, and $C_0A_0 = 15$. $\triangle A_1B_1C_1$ is inscribed in the incircle of $\triangle A_0B_0C_0$ such that it is similar to the first triangle. Beginning with $\triangle A_1B_1C_1$, the same steps are repeated to construct $\triangle A_2B_2C_2$, and so on infinitely many times. What is the value of $\sum_{i=0}^{\infty} A_iB_i$?

Answer: $\frac{845}{33}$

Solution: The area of a 13-14-15 triangle is 84 and its circumradius is $R = \frac{13\cdot14\cdot15}{2\cdot84} = \frac{65}{8}$. The semiperimeter is 21, so its inradius is $r = \frac{84}{21} = 4$. The ratio of the side lengths of $\triangle A_{i+1}B_{i+1}C_{i+1}$ to the side lengths of $\triangle A_iB_iC_i$ is then $\frac{r}{R} = \frac{32}{65}$. We are given that $A_0B_0 = 13$, so the sum is $13 \cdot \sum_{i=0}^{\infty} \left(\frac{32}{65}\right)^i = 13 \cdot \frac{1}{1-\frac{32}{65}} = 13 \cdot \frac{65}{33} = \boxed{\frac{845}{33}}$.

5. Let $ABCD$ be a square of side length 1, and let E and F be on the lines AB and AD, respectively, so that B lies between A and E, and D lies between A and F. Suppose that $\angle BCE = 20°$ and $\angle DCF = 25°$. Find the area of triangle ΔEAF.

Answer: 1

Solution: Since ΔEAF is a right triangle with a right angle at A, its area is $\frac{1}{2}(AE)(AF)$. Notice that $AE = AB + BE = 1 + BE$, and since ΔEBC is a right triangle, we have $BE = BC\tan(\angle BCE) = \tan(20°)$, so $AE = 1 + \tan(20°)$. Similarly, $AF = 1 + \tan(25°)$. Thus

$$\begin{aligned} \text{Area} &= \frac{1}{2}(AE)(AF) = \frac{1}{2}(1+\tan(20°))(1+\tan(25°)) \\ &= \frac{1}{2}(1+\tan(20°)+\tan(25°)+\tan(20°)\tan(25°)). \end{aligned}$$

By the tangent addition formula, we have $\tan(45°) = \frac{\tan(20°)+\tan(25°)}{1-\tan(20°)\tan(25°)}$, and since $\tan(45°) = 1$, it follows that $\tan(20°) + \tan(25°) = 1 - \tan(20°)\tan(25°)$. Thus

$$\text{Area} = \frac{1}{2}(1 + (1 - \tan(20°)\tan(25°)) + \tan(20°)\tan(25°)) = \boxed{1}.$$

6. $\odot A$, centered at point A, has radius 14 and $\odot B$, centered at point B, has radius 15. $AB = 13$. The circles intersect at points C and D. Let E be a point on $\odot A$, and F be the point where line EC intersects $\odot B$ again. Let the midpoints of DE and DF be M and N, respectively. Lines AM and BN intersect at point G. If point E is allowed to move freely on $\odot A$, what is the radius of the locus of G?

Answer: $\frac{65}{8}$

Solution: In $\odot A$ let minor 6.0pt$\overset{\frown}{CD}$ have measure $x°$, and in $\odot B$ let minor 6.0pt$\overset{\frown}{CD}$ have measure $y°$. In $\triangle DEF$, $\angle DEF = \frac{x}{2}°$ and $\angle DFE = \frac{y}{2}°$, so $\angle EDF = 180° - \frac{x}{2}° - \frac{y}{2}°$. In $\triangle ACD$, $\angle ADC = 90° - \frac{x}{2}°$, and in $\triangle BCD$, $\angle BDC = 90° - \frac{y}{2}°$. So, $\angle ADB = 180° - \frac{x}{2}° - \frac{y}{2}° = \angle EDF$. Quadrilateral $DMGN$ is cyclic since $\angle GMD + \angle GBD = 90° + 90° = 180°$, so $\angle MGN = 180° - \angle MDN = 180° - \angle ADB \Rightarrow \angle AGB = 180° - \angle ADB$.

Thus, $DAGB$ is also cyclic. This means that point G always lies on the circumcircle of $\triangle ADB$. So, we need to find the circumradius of a 13-14-15 triangle. Using Heron's Formula, the area of the triangle is $\sqrt{\left(\frac{13+14+15}{2}\right)\left(\frac{-13+14+15}{2}\right)\left(\frac{13-14+15}{2}\right)\left(\frac{13+14-15}{2}\right)} = 84$. The circumradius is $\frac{(AB)(AD)(BD)}{4[ABD]} = \frac{(13)(14)(15)}{4(84)} = \boxed{\frac{65}{8}}$.

7. An n-sided regular polygon with side length 1 is rotated by $\frac{180}{n}^\circ$ about its center. The intersection points of the original polygon and the rotated polygon are the vertices of a $2n$-sided regular polygon with side length $\frac{1-\tan^2 10^\circ}{2}$. What is the value of n?

Answer: 9

Solution: Let us call the center of the polygons O. Consider one of the intersection points of the original polygon and the rotated polygon, which we denote I. Denote the perpendicular foot of the center of the original polygon to the side of the original polygon that I lies on M, and denote the vertex of the original polygon closest to I as N. The length of MN is $\frac{1}{2}$, and $\angle MON$ is $\frac{180}{n}^\circ$, so the length of OM is $\frac{1/2}{\tan \frac{180}{n}^\circ}$. Also we have that $\angle MOI$ is $\frac{90}{n}^\circ$, so the length of MI is $OM \tan \frac{90}{n}^\circ = \frac{\frac{1}{2}\tan \frac{90}{n}^\circ}{\tan \frac{180}{n}^\circ}$. Using the double angle formula, we have $\frac{180}{n}^\circ = \frac{2\tan \frac{90}{n}^\circ}{1-\tan^2 \frac{90}{n}^\circ}$, so $\frac{\frac{1}{2}\tan \frac{90}{n}^\circ}{\tan \frac{180}{n}^\circ} = \frac{1-\tan^2 \frac{90}{n}^\circ}{4}$. Note that the side length of the $2n$-sided polygon is $2MI$, so we get $\frac{1-\tan^2 \frac{90}{n}^\circ}{2}$, which means that n should be $\boxed{9}$.

8. In triangle $\triangle ABC$, $AB = 5$, $BC = 7$, and $CA = 8$. Let E and F be the feet of the altitudes from B and C, respectively, and let M be the midpoint of BC. The area of triangle MEF can be expressed as $\frac{a\sqrt{b}}{c}$ for positive integers a, b, and c such that the greatest common divisor of a and c is 1 and b is not divisible by the square of any prime. Compute $a + b + c$.

Answer: 68

Solution: We first observe that quadrilateral $EFBC$ is cyclic with circumcenter M since $\angle BEC = \angle CFB = 90^\circ$. Thus, $MB = MF = ME = MC = BC/2 = 7/2$ as these segments are radii of the circumscribed circle of $EFBC$, so triangles $\triangle MBF$, $\triangle MEC$, and $\triangle MEF$ are isosceles.

From these observations, we deduce that $\angle BFM = \angle B$ and $\angle CEM = \angle C$, so $\angle BMF = 180^\circ - 2\angle B$ and $\angle CME = 180^\circ - 2\angle C$. Therefore,

$$\begin{aligned}\angle EMF &= 180^\circ - (180^\circ - 2\angle B + 180^\circ - 2\angle C)\\ &= 2(\angle B + \angle C) - 180^\circ\\ &= 2(180^\circ - \angle A) - 180^\circ\\ &= 180^\circ - 2\angle A.\end{aligned}$$

Now, by the Law of Cosines, we calculate

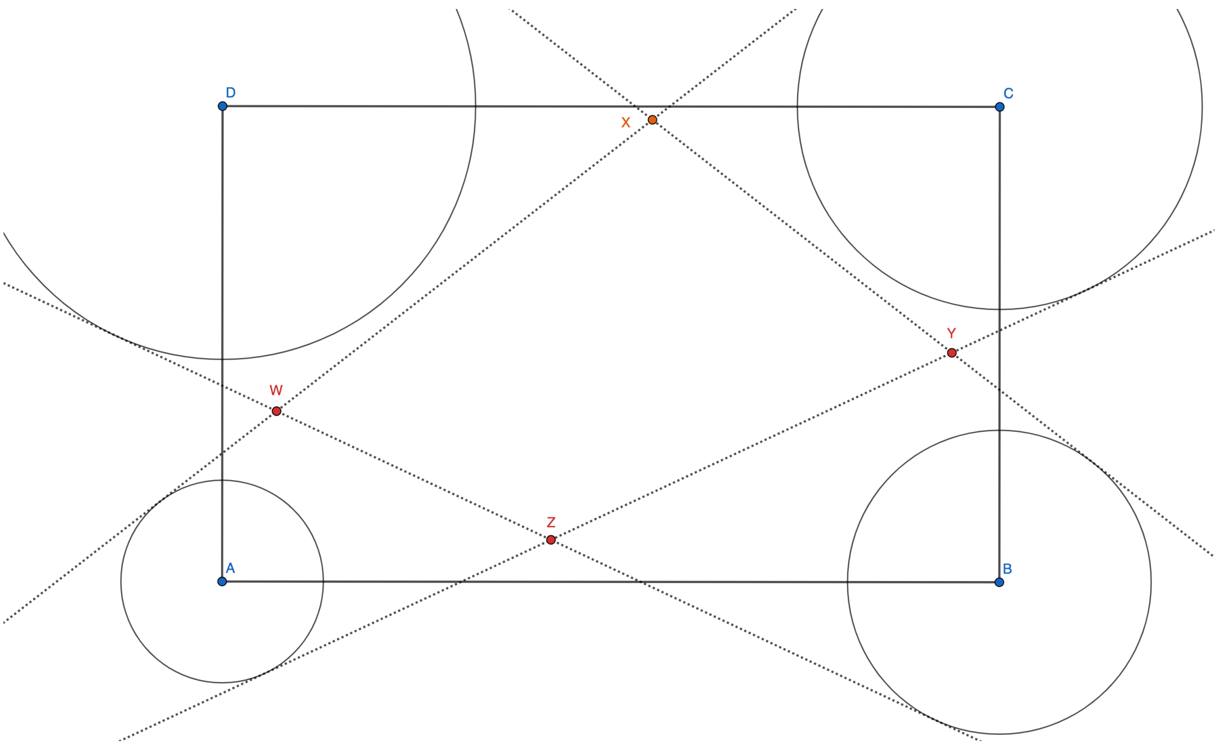
$$\cos A = \frac{AB^2 + AC^2 - BC^2}{2(AB)(AC)} = \frac{5^2 + 7^2 - 8^2}{2(5)(8)} = \frac{25 + 64 - 49}{80} = \frac{40}{80} = \frac{1}{2},$$

so $\angle A = 60°$ and $\angle EMF = 180° - 2(60°) = 180° - 120° = 60°$ and $\triangle MEF$ is, in fact, equilateral, with $ME = EF = FM = \frac{7}{2}$. Hence,

$$[MEF] = \frac{\left(\frac{7}{2}\right)^2 \sqrt{3}}{4} = \frac{49\sqrt{3}}{16},$$

so $a + b + c = 49 + 3 + 16 = 68$.

9. Rectangle $ABCD$ has an area of 30. Four circles of radius $r_1 = 2, r_2 = 3, r_3 = 5$, and $r_4 = 4$ are centered on the four vertices A, B, C, and D respectively. Two pairs of external tangents are drawn for the circles at A and C and for the circles at B and D. These four tangents intersect to form a quadrilateral $WXYZ$ where $\overline{WX}$ and $\overline{YZ}$ lie on the tangents through the circles on A and C. If $\overline{WX} + \overline{YZ} = 20$, find the area of quadrilateral $WXYZ$.



Answer: 70

Solution: We claim that $WXYZ$ is a circumscribed quadrilateral, or a tangential quadrilateral. To show this, note that the center of the rectangle is equidistant from each pair of external tangents with distance $\frac{r+R}{2}$ where r and R are the radii of opposing circles. Since $r_1 + r_3 = r_2 + r_4$, the center of the rectangle is equidistant from all four tangents. Therefore, a circle of radius $\frac{r_1+r_3}{2} = \frac{7}{2}$ can be inscribed in $WXYZ$.

The Pitot Theorem states that $WX + YZ = XY + ZW$ for any circumscribed quadrilateral. Thus, the perimeter of $WXYZ$ is $20 + 20 = 40$. It is not difficult to see that the area of a circumscribed quadrilateral is just sr where s is the semiperimeter and r is the radius of the inscribed circle. Our answer is then $\frac{40}{2} \cdot \frac{7}{2} = 70$.

Note that the area of $ABCD$ was only invoked so that there existed a quadrilateral $WXYZ$ with $\overline{WX} + \overline{YZ} = 20$.

10. In acute $\triangle ABC$, let points D, E, and F be the feet of the altitudes of the triangle from A, B, and C, respectively. The area of $\triangle AEF$ is 1, the area of $\triangle CDE$ is 2, and the area of $\triangle BFD$ is $2-\sqrt{3}$. What is the area of $\triangle DEF$?

Answer: $\mathbf{\sqrt{3}-1}$

Solution: In right $\triangle BEA$, we have $AE = AB\cos(\angle A)$, and in right $\triangle CFA$, we have $AF = AC\cos(\angle A)$. So, $\triangle AEF \sim \triangle ABC$ by SAS similarity. It follows that $EF = BC\cos(\angle A)$. Similarly, $\triangle DBF \sim \triangle ABC$ and $\triangle DEC \sim \triangle ABC$. Also, $DF = AC\cos(\angle B)$ and $DE = AB\cos(\angle C)$.

From the similar triangles, $\angle AFE = \angle C$ and $\angle BFD = \angle C$, so $\angle EFD = 180^\circ - 2\angle C$. Using the law of sines,

$[DEF] = \frac{1}{2}(FD)(FE)\sin(\angle EFD)$

$= \frac{1}{2}(AC\cos(\angle B))(BC\cos(\angle A))\sin(180^\circ - 2\angle C)$

$= \frac{1}{2}(AC)\cos(\angle B)(BC)\cos(\angle A)\sin(2\angle C)$

$= (AC)\cos(\angle B)(BC)\cos(\angle A)\sin(\angle C)\cos(\angle C)$

$= 2(\frac{1}{2}(AC)(BC)\sin(\angle C))\cos(\angle A)\cos(\angle B)\cos(\angle C)$

$= 2[ABC]\cos(\angle A)\cos(\angle B)\cos(\angle C)$.

Also from the similar triangles, $[AEF] = [ABC]\cos^2(\angle A)$, $[DBF] = [ABC]\cos^2(\angle B)$, and $[DEC] = [ABC]\cos^2(\angle C)$.

Now we have

$2[ABC]\cos(\angle A)\cos(\angle B)\cos(\angle C)$

$= [DEF] = [ABC] - [AEF] - [DBF] - [DEC]$

$= [ABC] - 5 + \sqrt{3}$

and

$[ABC]\cos^2(\angle A)[ABC]\cos^2(\angle B)[ABC]\cos^2(\angle C)$

$= [AEF][DBF][DEC]$

$= 4 - 2\sqrt{3}$.

Let $[ABC] = x$ and $\cos(\angle A)\cos(\angle B)\cos(\angle C) = y$. Our two equations are $2xy = x - 5 + \sqrt{3}$ and $x^3y^2 = 4 - 2\sqrt{3}$. From the first equation, $y = \frac{x-5+\sqrt{3}}{2x}$. Substituting this into the second equation and simplifying gives

$$x(x - 5 + \sqrt{3})^2 = 16 - 8\sqrt{3}.$$

Notice that $16 - 8\sqrt{3} = 4(4 - 2\sqrt{3}) = 4(\sqrt{3} - 1)^2$. So, $x = 4$ is a solution to the equation. If we write the equation as a cubic and factor out $(x - 4)$, we see that 4 is the only solution that is greater than $[AEF] + [DBF] + [DEC] = 5 - \sqrt{3}$. Thus, we have $[ABC] = 4$. The area of $\triangle DEF$ is $4 - 1 - 2 - (2 - \sqrt{3}) = \boxed{\sqrt{3} - 1}$.