1. Find all integer solutions to
$$\frac{1}{\log_8 n} + \frac{1}{\log_n \frac{1}{4}} = -\frac{5}{2}$$

Answer: 64

Solution: Applying $\log_a b = \frac{\log_c b}{\log_c a}$ with $c = 2$ on both fractions, this becomes:
$$\frac{\log_2 8}{\log_2 n} + \frac{\log_2 n}{\log_2 \frac{1}{4}} = -\frac{5}{2}$$
Substituting $x = \log_2 n$ and evaluating $\log_2 8 = 3$ and $\log_2 \frac{1}{4} = -2$, this becomes:
$$\frac{3}{x} + \frac{x}{-2} = -\frac{5}{2}$$
Multiplying both sides by $-2x$ yields $-6 + x^2 = 5x$, which factors into $(x-6)(x+1) = 0$. This quadratic has solutions $x = 6$ and $x = -1$. When $x = -1$, we have $\log_2 n = -1 \implies n = \frac{1}{2}$, which is not an integer. When $x = 6$, we have $\log_2 n = 6 \implies n = \boxed{64}$.

2. Let $f : \mathbb{N} \to \mathbb{N}$ be a bijection such that for all a, either $a = f(a)^2$ or $f(a) = a^2$. For how many numbers less than 40 is $f(a) \neq a^2$?

Answer: 4

Solution: We start by figuring out the pattern. We can see that $f(0) = 0$ and $f(1) = 1$. Then $f(2) = 4$ and $f(3) = 4$. Then we can see that $f(4) = 2$ or else no other number satisfies $f(a) = 2$ and this is a bijection. Similarly $f(9) = 3$. Then $f(16) = 16^2$ since $f(2) = 4$ and we have a bijection. So, we can see that the only numbers where $f(a) \neq a^2$ are $4, 9, 25, 36$. So, there are $\boxed{4}$ numbers.

3. Let x be a real number satisfying the equation $x^2 - 3x + 1 = 0$. Then $x^{16} - kx^8 + 1 = 0$ for some constant k. Compute k.

Answer: 2207.

Solution: Rearrange the given quadratic equation to $x + \frac{1}{x} = 3$. Letting $P_n(x) = x^n + \frac{1}{x^n}$, it is easy to derive the recursive formula
$$P_{2n}(x) = P_n(x)^2 - 2.$$
We know that $P_1(x) = 3$, so we can apply the recursion to obtain:
$$\begin{aligned} P_2(x) &= P_1(x)^2 - 2 = 7, \\ P_4(x) &= P_2(x)^2 - 2 = 47, \\ P_8(x) &= P_4(x)^2 - 2 = 2207. \end{aligned}$$
Hence, $x^8 + \frac{1}{x^8} = 2207$, and rearranging yields $x^{16} - 2207x^8 + 1 = 0$. Thus, $k = 2207$.