1. How many nonnegative integers less than 2019 are not solutions to $x^8 + 4x^6 - x^2 + 3 \equiv 0 \pmod 7$?

Answer: 289

Solution: We know that x is a solution to $x^8 + 4x^6 - x^2 + 3 \equiv 0 \pmod 7$ if $x \pmod 7$ is. Thus, we only need to test the numbers 0 through 6. Alternatively, we can factor our polynomial as

$$x^8 + 4x^6 - x^2 + 3 \equiv x^8 + 4x^6 - x^2 - 4 \equiv (x^6 - 1)(x^2 + 4) \pmod 7$$

Then since 7 is prime, by Fermat's Little Theorem, $1, 2, 3, 4, 5, 6$ all satisfy $x^6 - 1$. Hence, the only numbers that are not solutions are multiples of 7. Thus, the number of multiples of 7 from 0 to 2019 is $1 + \lfloor \frac{2019}{7} \rfloor = \boxed{289}$.

2. How many rational numbers can be written in the form $\frac{a}{b}$ such that a and b are relatively prime positive integers and the product of a and b is $(25!)$?

Answer: 512

Solution: From $\gcd(a, b) = 1$, we note that for each prime p dividing $25!$, either $p|a$ or $p|b$. As a result, for each prime p dividing $25!$, we have two ways to choose which of a or b it divides. Since there are nine primes less than 25, the answer is $2^9 = \boxed{512}$.

3. Connie finds a whiteboard that has magnet letters spelling $MISSISSIPPI$ on it. She can rearrange the letters, in which identical letters are indistinguishable. If she uses all the letters and does not want to place any Is next to each other, how many distinct rearrangements are possible?

Answer: 7350

Solution: Imagine that each I is a separator, which splits the word into 5 sections. First, we find the number of ways to put letters in each bucket supposing that all the letters are the same. Since none of the Is can be next to each other, the middle 3 buckets must contain at least one letter. The remaining 4 letters can go into any of the 5 buckets, and there are $\binom{4+5-1}{5-1} = \binom{8}{4} = 70$ to do that. Next, we need to find the number of ways to order the non-I letters. There are 7 letters total, with 4 Ss, 2 Ps, and 1 M, so there are $\frac{7!}{4!2!1!} = 105$ ways to order them. Therefore, the answer is $70 \cdot 105 = \boxed{7350}$.

4. In your drawer you have two red socks and a blue sock. You randomly select socks, without replacement, from the drawer. However, every time you take a sock, another blue sock magically appears in the drawer. You stop taking socks when you have a pair of red socks. At this time, say you have x socks total. What is the expected value of x?

Answer: $\frac{9}{2}$

Solution: Let b be the expected number of socks we draw until we have two red socks, given that we currently have one red sock. If we have one red sock already, the drawer must contain two blue socks and one red sock. Thus, we can write $b = \frac{2}{3}(1 + b) + \frac{1}{3}$ because there is a $\frac{2}{3}$ probability we draw another blue sock and a $\frac{1}{3}$ probability we get the final red sock. Solving for b, we get $b = 3$.

Let a be the expected number of socks we draw until we have two red socks, given that we currently have no red socks. Then the drawer must contain two red socks and one blue sock. We then have $a = \frac{1}{3}(1 + a) + \frac{2}{3}(1 + b)$ because there is a $\frac{1}{3}$ probability of drawing a blue sock and a $\frac{2}{3}$ probability we get a red sock. Solving for a, we get $a = \boxed{\frac{9}{2}}$.

Solution: Alternatively, recall that expected value is the sum of the values weighted by their probability. So, consider if it takes n socks to get the red pair. We note that $n \geq 2$ since a pair is two socks. Then we can see the probability it takes n socks is:

$$\left(\frac{1}{3}\right)^{n-2} \cdot \frac{2}{3} \cdot \frac{1}{3} + \left(\frac{1}{3}\right)^{n-3} \cdot \frac{2}{3} \cdot \left(\frac{2}{3}\right)^{1} \frac{1}{3} + ... + \frac{2}{3} \cdot \left(\frac{2}{3}\right)^{n-2} \frac{1}{3} = \frac{2}{3^n} \cdot \sum_{i=0}^{n-2} 2^i = \frac{2^n - 2}{3^n}$$

Now we calculate the expected value:

$$\begin{aligned}
E &= \sum_{n=2}^{\infty} n \cdot \frac{2^n - 2}{3^n} \\
&= 2 \cdot \sum_{n=2}^{\infty} \frac{2^n - 2}{3^n} + \sum_{n=3}^{\infty} \frac{2^n - 2}{3^n} + \sum_{n=4}^{\infty} \frac{2^n - 2}{3^n} + ... \\
&= 2 \cdot \left(\frac{4/9}{1-2/3} - \frac{2/9}{1-1/3}\right) + \frac{8/27}{1-2/3} - \frac{2/27}{1-1/3} + \frac{16/81}{1-2/3} - \frac{2/81}{1-1/3} + ... \\
&= 2 \cdot \left(\frac{4}{3} - \frac{1}{3}\right) + 3 \cdot \frac{8/27}{1-2/3} - \frac{3}{2} \cdot \frac{2/27}{1-1/3} \\
&= 2 + \frac{8}{3} - \frac{1}{6} \\
&= 2 + 5/2 = \boxed{\frac{9}{2}}.
\end{aligned}$$

5. Let $S(n)$ denote the sum of the digits of positive integers n. For some positive integer k, it is known that $S(k) = 152$ and that $S(k+1)$ is a multiple of 5. What is the difference between the largest and smallest possible values of $S(k+1)$?

Answer: 90

Solution: Let us denote

$$k = \sum_{i=0}^{n} 10^i a_i, \qquad k+1 = \sum_{i=0}^{n'} 10^i a'_i$$

where $a_j, a'_j \in \{0, 1, ..., 9\}$, and $n+1$ and $n'+1$ are the number of digits in k and $k+1$ respectively. Consider the case where the last m digits of k be all equal to 9, i.e. $a_0 = a_1 = ... = a_{m-1} = 9$, and $a_m < 9$. Then we have $a'_0 = a'_1 = ... = a'_{m-1} = 0, a'_m = a_m + 1$. This gives us $S(k+1) = S(k) - 9m + 1 = 153 - 9m$. Since these cases span all possible values of k, $S(k+1)$ is necessarily a value of this form. Using the fact that $5|S(k+1)$, we identify that the largest possible value of $S(k+1)$ is 135 ($m = 2$), and the smallest possible value is 45 ($m = 12$). Thus, we have the difference between the largest and smallest possible values is $135 - 45 = \boxed{90}$.

6. The numbers $1, 2, ..., 13$ are written down, one at a time, in a random order. What is the probability that at no time during this process the sum of all written numbers is divisible by 3?

Answer: $\mathbf{\frac{1}{182}}$

Solution: Since we only care about divisibility by 3, we can consider this list mod 3, i.e. $\{1, 2, 0, ..., 1\}$. With the exception of the first slot, the 0 mod 3 elements can be inserted anywhere without changing the partial sum mod 3. Using this list, it is clear that the only way the other elements can be ordered is $\{1, 1, 2, 1, 2, 1, 2, 1, 2\}$. There are $5!4!$ ways to arrange the 5 elements that are 1 mod 3 and the 4 elements that are 2 mod 3 in this manner. Then, there are 9 spots to insert "3" in this list, 10 for "6", and so on. Thus, the probability that any given arrangement is of the desired form is

$$\frac{5!4! \cdot 9 \cdot 10 \cdot 11 \cdot 12}{13!} = \boxed{\frac{1}{182}}$$

7. Let $S = 1 + 2 + 3 + ... + 100$. Find $(100!/4!) \mod S$.

Answer: 2950

Solution: First note that $S = 50 \cdot 101$ and $100!/4! = 50 \cdot 99!/12$, so $(100!/4!) \bmod S$ is equivalent to 50 times the value of $(99!/12) \bmod 101$. Since 101 is prime, Wilson's Theorem gives us that $100! \equiv -1 \pmod{101}$, so $99! \equiv 1 \pmod{101}$. As a result, it suffices to determine the inverse of 12 mod 101. To solve $12x \equiv 1 \pmod{101}$, we can write $12x = 1 + 101n$ for some positive integer n. Taking the equation mod 12, we get $5n \equiv -1 \pmod{12}$, so $n \equiv 7 \pmod{12}$. Then the inverse of 12 mod 101 is $x = \frac{1+101\cdot 7}{12} = 59$. Hence, the answer is $50 \times 59 = \boxed{2950}$.

8. Let $S_n = \sum_{j=1}^{n} j^3$. Find the smallest positive integer n greater than 100 such that the first three digits of S_n are 100.

Answer: 141

Solution: We know that $S_n = \sum_{j=1}^{n} j^3 = (\frac{n(n+1)}{2})^2$ and we want the smallest positive integer n such that S_n begins in 100. Let $(n(n+1)/2)^2 \geq 10^k$ for some positive integer k. Rearranging, we get $n^2 + n \geq 2 \cdot 10^{k/2}$, and solving for n gives $n \geq \frac{1}{2}(-1 + \sqrt{8 \cdot 10^{k/2}}) = \sqrt{2 \cdot 10^{k/2}} - 1/2$. Additionally, $n \geq 100$ which means that $k \geq 8$. Using this value, the smallest possible n is $n \geq \sqrt{2 \cdot 10^4} - 1/2 \geq 140$. It is clear that $n = 140$ does not work, but $S_{141} = (141 \cdot 71)^2 = 10011^2 = 100220121$. Therefore, the smallest possible value is $n = \boxed{141}$.

9. Edward has a 3×3 tic-tac-toe board and wishes to color the squares using 3 colors. How many ways can he color the board such that there is at least one row whose squares have the same color and at least one column whose squares have the same color? A coloring does not have to contain all three colors and Edward cannot rotate or reflect his board.

Answer: 1785

Solution: Note that any row and any column intersects at one square, so if there exists a row whose squares are all the same color and a column whose squares are all the same color, that color must be the same. Let S_{ij} be the set of colorings having i rows and j columns the same color. We wish to compute $\left|\bigcup_{i=1}^{3} \bigcup_{j=1}^{3} S_{ij}\right|$.

This can be computed using the Principle of Inclusion-Exclusion. The total number of colorings with at least i rows and at least j columns the same color is $3\binom{3}{i}\binom{3}{j}3^{(3-i)(3-j)}$ because there are 3 ways to choose which color those rows and columns will be, $\binom{3}{i}$ ways to choose the rows, $\binom{3}{j}$ ways to choose the columns, and $3^{(3-i)(3-j)}$ ways to arbitrarily color the rest of the $(3-i)(3-j)$ squares. Hence, our answer is

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j=1}^{3}(-1)^{i+j}3\binom{3}{i}\binom{3}{j}3^{(3-i)(3-j)} &= 3\sum_{i=1}^{3}\binom{3}{i}(-1)^i\sum_{j=1}^{3}(-1)^j\binom{3}{j}(3^{3-i})^{3-j} \\ &= 3\sum_{i=1}^{3}\binom{3}{i}(-1)^i\left((3^{3-i}-1)^3-(3^{3-i})^3\right)\end{aligned}$$

Computing the summation, we get that the total number of such colorings is $\boxed{1785}$.

10. Let N be a positive integer that is a product of two primes p, q such that $p \le q$ and for all a, $a^{5N} \equiv a \mod 5N$. Find the sum of p over all possible values N.

Answer: 59

Solution: Note that

$$2^{5pq} = 2 \pmod 5 \implies 2^{pq} = 2 \pmod 5 \implies 2^{pq-1} = 1 \mod 5$$

So, we have that $\mathrm{ord}_5(2)$ must divide $pq-1$ by the last equation. So, we know that p, q are odd. Now let α be a primitive root $\pmod p$ and then $\alpha^{5pq-1} \equiv 1 \pmod p$. So, $p-1 \mid 5pq-1$, and $p-1 \mid 5pq-1-5q(p-1) = 5q-1$. We can similarly see that $q-1 \mid 5p-1$. If $p = q$, then $p = q = 5$ but $6^{125} \equiv 1 \pmod{125}$, a contradiction. So, WLOG $p < q$ and $q \ge p+2$. So,

$$1 < \frac{5p-1}{q-1} \le \frac{5q-1}{p+1} < 5$$

Then we do case work on $\frac{5p-1}{q-1}$ is $2, 3$ or 4.

Cases:

(a) $(5p-1)/(q-1) = 2$: Then $2q = 5p+1$. But $p-1|5q-1 = 5(5p+1)/2-1 = 25p/2+3/2 = 1/2(25p+3)$. So $p-1$ needs to divide $(25p+3)-(25p-25) = 28$. Hence, the options are $p-1 = 2, 4, 7, 14, 28$. Since p is prime, have $p = 3, 5, 29$. Solving for prime q, we see that the only solutions are $p = 5, q = 13$ and $p = 29, q = 73$.

(b) $(5p-1)/(q-1) = 3$ implies $3q = 5p+2$. But $p-1|5q-1 = 5(5p+2)/3-1 = 1/3(25p+7)$. So, $p-1$ divides $(25p+7)-(25p-25) = 32$ giving options $p-1 = 2, 4, 8, 16, 32$. Since p is prime, we have $p = 3, 5, 17$. Solving for prime q, we see that the only solution is $p = 17, q = 29$.

(c) $(5p-1)/(q-1) = 4$ gives $4q = 5p+3$. But $p-1|5q-1 = 5(5p+3)/4-1 = 1/4(25p+11)$. So, $p-1$ also divides $(25p+11)-(25p-25) = 36$. So, $p-1$ can be $2, 3, 4, 6, 9, 12, 18, 36$. Since p is prime, we have $p = 3, 5, 7, 13, 19, 37$. Solving for prime q, we get $p = 5, q = 7$, $p = 13, q = 17$, $p = 37, q = 47$.

We also have the condition that $\mathrm{ord}_5(2)|pq-1$, and since $\mathrm{ord}_5(2) = 4$, we have that $pq \equiv 1 \pmod 4$. Also, we note that if N is 65 then $2^{5N} \equiv 132 \mod 5N$. The remaining values of p have the sum of

$$29 + 17 + 13 = \boxed{59}$$