1. David owns a parking lot for vehicles. A vehicle is either a motorcycle with two wheels or a car with four wheels. Today, there are 100 vehicles parked in his parking lot. The total number of wheels in David's parking lot is 326. If David collects \$1.00 from each motorcycle and \$2.00 from each car per day, how much money in dollars does David collect today?

   Answer: 163

   Solution: Let x and y be the number of motorcycles and cars, respectively. We have

$$x + y = 100$$

$$2x + 4y = 326.$$

   But note that the amount collected will be $x + 2y$ which is half of $2x + 4y$. So, David can collect $x + 2y = \frac{326}{2} = \boxed{163}$ dollars.

2. Three consecutive terms of a geometric sequence of positive integers multiply to $1,000,000$. If the common ratio is greater than 1, what is the smallest possible sum of the three terms?

   Answer: 305

   Solution: By the AM-GM inequality, the sum is minimized when the terms are as close as possible to each other. So, we want r to be as close to 1 as possible. First, we find the middle term. If the first term is a, then the three terms are a, ar, ar^2 respectively. So, $(ar)^3 = 1,000,000$. Thus, the middle term can be computed by taking the cube root of $1,000,000$, which is 100. Next, we must find the 2 factors of $\frac{1,000,000}{100} = 10,000$ which are closest to 100. Inspection reveals that 80 and 125 are the 2 desired factors of $10,000$, hence our sum is $80 + 100 + 125 = \boxed{305}$. Moreover, we can confirm that no other possible factors exist, as no numbers between 80 and 100 exclusive or between 100 and 125 exclusive have only 2 or 5 as prime factors.

3. Let x, y be real numbers such that

$$\begin{aligned} x + y &= 2, \\ x^4 + y^4 &= 1234. \end{aligned}$$

   Find xy.

   Answer: -21

   Solution: Note that:

$$x^4 + y^4 = (x + y)^4 - 4xy(x + y)^2 + 2(xy)^2$$

   Let $P = xy$ be the product we want to solve for. Then the equation $x^4 + y^4 = 1234$ becomes:

$$\begin{aligned} 1234 &= 16 - 16P + 2P^2 \\ \implies P^2 - 8P - 609 &= 0 \\ \implies (P - 29)(P + 21) &= 0. \end{aligned}$$

   It follows that $P = 29$ or $P = -21$. If P is 29, then x and y are the roots of the quadratic $X^2 - 2X + 29$, which are not real. Hence $P = \boxed{-21}$.

4. Evaluate $(350 + 90\sqrt{15})^{\frac{1}{3}} + (350 - 90\sqrt{15})^{\frac{1}{3}}$.

   Answer: 10

Solution: Let $S = (350 + 90\sqrt{15})^{\frac{1}{3}} + (350 - 90\sqrt{15})^{\frac{1}{3}}$. Then,

$$\begin{aligned}S^3 &= 350 + 90\sqrt{15} + 3(350 + 90\sqrt{15})^{\frac{1}{3}}(350 - 90\sqrt{15})^{\frac{1}{3}}((350 + 90\sqrt{15})^{\frac{1}{3}} + (350 - 90\sqrt{15})^{\frac{1}{3}}) \\ &\quad + 350 - 90\sqrt{15} \\ &= 700 + 3(350^2 - (90\sqrt{15})^2)^{\frac{1}{3}} S \\ &= 700 + 30S.\end{aligned}$$

Note that both $(350 + 90\sqrt{15})$ and $(350 - 90\sqrt{15})$ are positive. Hence, S must be a positive number that satisfies $S^3 - 30S - 700 = (S - 10)(S^2 + 10S + 70) = 0$. In fact, $S = 10$ is the only real solution to the cubic equation. Therefore, $(350+90\sqrt{15})^{\frac{1}{3}}+(350-90\sqrt{15})^{\frac{1}{3}} = \boxed{10}$.

5. Let $f(x) = 36x^4 - 36x^3 - x^2 + 9x - 2$. Then let the four roots of $f(x)$ be r_1, r_2, r_3, and r_4. Find the value of

$$(r_1 + r_2 + r_3)(r_1 + r_2 + r_4)(r_1 + r_3 + r_4)(r_2 + r_3 + r_4).$$

Answer: $\mathbf{\frac{1}{6}}$

Solution 1: Note that $s = r_1 + r_2 + r_3 + r_4 = \frac{36}{36} = 1$. Then

$$\begin{aligned}(r_1 + r_2 + r_3)(r_1 + r_2 + r_4)(r_1 + r_3 + r_4)(r_2 + r_3 + r_4) &= (s - r_4)(s - r_3)(s - r_2)(s - r_1) \\ &= \frac{f(s)}{36} \\ &= \frac{36 - 36 - 1 + 9 - 2}{36} \\ &= \boxed{\frac{1}{6}}.\end{aligned}$$

Solution 2: Note that $s = r_1 + r_2 + r_3 + r_4 = \frac{36}{36} = 1$ and also $f(x) = (3x - 1)(3x - 2)(2x - 1)(2x + 1)$. So, the roots are $\frac{1}{3}, \frac{2}{3}, \frac{1}{2}, -\frac{1}{2}$. Hence,

$$\begin{aligned}(r_1 + r_2 + r_3)(r_1 + r_2 + r_4)(r_1 + r_3 + r_4)(r_2 + r_3 + r_4) &= (s - r_4)(s - r_3)(s - r_2)(s - r_1) \\ &= \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) \\ &= \boxed{\frac{1}{6}}.\end{aligned}$$

6. Let $f(X)$ be a complex monic quadratic with real roots $\frac{1}{3}, \frac{2}{3}$. (The polynomial $f(X)$ is of the form $X^2 + bX + c$ where b, c, X are complex numbers.) If $|z| = 1$, what is the sum of all possible values of $f(z)$ such that $f(z) = \overline{f(z)}$?

Answer: $\mathbf{\frac{5}{3}}$

Solution: We know that $f(X) = (X - \frac{1}{3})(X - \frac{2}{3})$. From now on, we refer to the roots as r_1, r_2. Then we know that $f(z)$ can be written as $(z - r_1)(z - r_2)$. Notice that by properties of conjugation, we can write $\overline{f(z)} = (\overline{z} - r_1)(\overline{z} - r_2)$. Expanding and simplifying $f(z) = \overline{f(z)}$ shows that

$$\overline{z}^2 - \overline{z}(r_1 + r_2) = z^2 - z(r_1 + r_2).$$

This can be rearranged as $(\overline{z} - z)(\overline{z} + z) = (\overline{z} - z)(r_1 + r_2)$.

If $\overline{z} = z$, then z has no imaginary part and thus z is equal to 1 or -1, yielding $f(1) = \frac{2}{9}$ or $f(-1) = \frac{20}{9}$. Otherwise, we can cancel $(\overline{z} - z)$. Separating the real and imaginary parts of z as $x + iy$, we see that $x = \frac{r_1+r_2}{2}$ and $y^2 = 1 - x^2$. Notice that $f(z) = \overline{f(z)}$ means $f(z)$ is real, hence we see that

$$f(z) = \mathrm{Re}((x+iy-r_1)(x+iy-r_2)) = x^2-y^2-x(r_1+r_2)+r_1r_2 = x^2-(1-x^2)-2x^2+r_1r_2 = -\frac{7}{9}.$$

The three options are then $\frac{2}{9}, \frac{20}{9}, -\frac{7}{9}$ and therefore, the sum of all possible values of $f(z)$ is $\boxed{\frac{5}{3}}$.

7. Given that x, y are real numbers satisfying $x > y > 0$, compute the minimum value of

$$\frac{5x^2 - 2xy + y^2}{x^2 - y^2}.$$

Answer: $\mathbf{2 + 2\sqrt{2}}$

Solution: We first use partial fraction decomposition on this function. Doing so gives us

$$\begin{aligned}\frac{5x^2 - 2xy + y^2}{x^2 - y^2} &= \frac{5x^2 - 2xy + y^2}{(x+y)(x-y)}\\ &= \frac{3x^2 - 2xy + 3y^2 + 2(x^2 - y^2)}{(x+y)(x-y)}\\ &= \frac{x^2 + 2xy + y^2 + 2(x^2 - 2xy - y^2)}{(x+y)(x-y)} + 2\\ &= \frac{(x+y)^2 + 2(x-y)^2}{(x+y)(x-y)} + 2\\ &= \frac{x+y}{x-y} + \frac{2(x-y)}{x+y} + 2.\end{aligned}$$

We can then apply AM-GM to the first two terms to get

$$\frac{x+y}{x-y} + \frac{2(x-y)}{x+y} \geq 2\sqrt{2}.$$

Thus, the minimum is $\boxed{2 + 2\sqrt{2}}$, which is achieved when x, y satisfy the equation $(x+y)^2 = 2(x-y)^2$, which has solutions $y = (3 \pm 2\sqrt{2})x$. When $y = (3 - 2\sqrt{2})x$ and $x > 0$, the condition $x > y > 0$ is satisfied.

8. The equation

$$(x-1)(x-2)(x-4)(x-5)(x-7)(x-8) = (x-3)(x-6)(x-9)$$

has distinct roots $r_1, r_2, \ldots, r_6$. Evaluate

$$\sum_{i=1}^{6}(r_i - 1)(r_i - 2)(r_i - 4).$$

Answer: 273

Solution: Let $P(x)$ be a generic 6th degree polynomial $x^6 - a_5x^5 + a_4x^4 - a_3x^3 + a_2x^2 - a_1x_1 + a_0$ with roots $r_1, r_2, \ldots, r_6$. Let s_1 be the sum of all roots, s_2 be the sum of the squares of all roots, and s_3 be the sum of the cubes of all roots. We use Vieta's formulas to find expressions for s_1, s_2, s_3 in terms of the coefficients a_i:

$$\begin{aligned}
a_5 &= s_1 \\
a_4 &= r_1r_2 + r_1r_3 + \cdots + r_4r_6 + r_5r_6 = \frac{1}{2}(s_1^2 - s_2) \\
a_3 &= r_1r_2r_3 + r_1r_2r_4 + \cdots + r_3r_5r_6 + r_4r_5r_6 \\
s_1s_2 &= (r_1^3 + r_2^3 + \cdots + r_6^3) + (r_1r_2^2 + r_1r_3^2 + \cdots + r_6r_4^2 + r_6r_5^2) \\
s_1s_2 - s_3 &= r_1r_2^2 + r_1r_3^2 + \cdots + r_6r_4^2 + r_6r_5^2 \\
s_1^3 &= (r_1^3 + r_2^3 + \cdots + r_6^3) + 3(r_1r_2^2 + r_1r_3^2 + \cdots + r_6r_5^2) \\
&\quad + 6(r_1r_2r_3 + r_1r_2r_4 + \cdots + r_4r_5r_6) \\
&= s_3 + 3(s_1s_2 - s_3) + 6a_3 \\
s_3 &= \frac{3}{2}s_1s_2 - \frac{1}{2}s_1^3 + 3a_3
\end{aligned}$$

Now let's compare $Q(x) = (x-1)(x-2)(x-4)(x-5)(x-7)(x-8) - (x-3)(x-6)(x-9)$ and $R(x) = (x-1)(x-2)(x-4)(x-5)(x-7)(x-8)$. Let the coefficients of $Q(x)$ be denoted by a_n and the coefficients of $R(x)$ be denoted by a'_n. Also define s'_1, s'_2, and s'_3 to be the sum of 1st, 2nd, and 3rd powers of roots in $R(x)$, and $r'_1, \ldots, r'_6$ to be the roots of $R(x)$ (which are just 1, 2, 4, 5, 7, 8).

We can see that $a_5 = a'_5$, $a_4 = a'_4$, $a_3 = a'_3 + 1$. Note that s_1 and s_2 for any sixth degree polynomial only depend on a_4 and a_5. This means $s_1 = s'_1, s_2 = s'_2$, and subsequently $s_3 = s'_3 + 3$. Now we find the sum in question:

$$\begin{aligned}
\sum_{i=1}^{6}(r_i - 1)(r_i - 2)(r_i - 4) &= s_3 + c_2s_2 + c_1s_1 + c_0 \\
&= 3 + s'_3 + c_2s'_2 + c_1s'_1 + c_0 \\
&= 3 + \sum_{i=1}^{6}(r'_i - 1)(r'_i - 2)(r'_i - 4).
\end{aligned}$$

Now, we simply need to compute this sum over the known roots 1, 2, 4, 5, 7, 8. The first 3 roots yield a value of zero inside the sum, so the answer is simply $3 + 4\cdot3\cdot1 + 6\cdot5\cdot3 + 7\cdot6\cdot5 = \boxed{273}$.

9. Suppose we have a strictly increasing function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ where $\mathbb{Z}^+$ denotes the set of positive integers. We also know that both

$$f(f(1)), f(f(2)), f(f(3)), \ldots$$

and

$$f(f(1)+1), f(f(2)+1), f(f(3)+1), \ldots$$

are arithmetic sequences. Given that $f(1) = 1$ and $f(2) = 3$, find the maximum value of

$$\sum_{j=1}^{100} f(j).$$

Answer: 10000

Solution: We claim that $f(1), f(2), f(3), f(4), \ldots$ is also an arithmetic sequence.

Since f is a strictly increasing function from positive integers to positive integers, $f(x+1) - f(x) \geq 1 = (x+1) - x$ for all $x \in \mathbb{Z}^+$. Thus, for all $y > x$, $f(y) - f(x) = f(y) - f(y-1) + f(y-1) + f(y-2) + \cdots + f(x+1) - f(x) \geq y - x$.

Consider the differences $d_n = f(n+1) - f(n)$. Note that $d_n = f(n+1) - f(n) \leq f(f(n+1)) - f(f(n))$ from above and $f(f(n+1)) - f(f(n)) = D_1$ is a constant since $f(f(1)), f(f(2)), f(f(3)), \ldots$ is an arithmetic sequence. Hence, d_n is bounded above by D_1. Also, since f is a strictly increasing sequence, $d_n = f(n+1) - f(n) > 0$ for all $n \in \mathbb{Z}^+$. Hence, $1 \leq d_n \leq D_1$ for all $n \in \mathbb{Z}^+$. Let m and M be the smallest and largest value of d_n over all $n \in \mathbb{Z}^+$.

Let the common difference of the second arithmetic sequence $f(f(1)+1), f(f(2)+1), f(f(3)+1), \ldots$ be D_2. Now observe that

$$\begin{aligned} d_{f(n+1)} - d_{f(n)} &= (f(f(n+1)+1) - f(f(n+1))) - (f(f(n)+1) - f(f(n))) \\ &= f(f(n+1)+1) - f(f(n)+1) - (f(f(n+1)) - f(f(n))) \\ &= D_2 - D_1. \end{aligned}$$

If $D_2 \neq D_1$, then the sequence $d_{f(1)}, d_{f(2)}, d_{f(3)}, \ldots$ is either strictly increasing to $+\infty$ or strictly decreasing to $-\infty$ which is impossible because d_n is bounded by 1 and D_1. Thus, $D_2 = D_1$ and $d_{f(n)}$ is a constant for all $n \in \mathbb{Z}^+$. From now on, call $D_1 = D_2 = D$.

Let k and K be such that $d_k = m$ and $d_K = M$. Hence,

$$\begin{aligned} D &= f(f(k+1)) - f(f(k)) \\ &= \sum_{j=f(k)}^{f(k+1)-1} (f(j+1) - f(j)) \\ &= d_{f(k+1)-1} + d_{f(k+1)-2} + \cdots + d_{f(k)} \\ &\leq mM \end{aligned}$$

since there are $f(k+1) - 1 - f(k) + 1 = d_k = m$ terms in the summation and each term is at most M. Similarly,

$$\begin{aligned} D &= f(f(K+1)) - f(f(K)) \\ &= \sum_{j=f(K)}^{f(K+1)-1} (f(j+1) - f(j)) \\ &= d_{f(K+1)-1} + d_{f(K+1)-2} + \cdots + d_{f(K)} \\ &\geq mM \end{aligned}$$

since there are $f(K+1) - 1 - f(K) + 1 = d_K = M$ terms in the summation and each term is at least m.

Thus, $mM \leq D \leq mM$ which means $D = mM$. For this to be achieved, $d_{f(k)} = (f(f(k)+1) - f(f(k))) = M$ and $d_{f(K)} = (f(f(K)+1) - f(f(K))) = m$. But $d_{f(n)}$ is a constant for all $n \in \mathbb{Z}^+$. Hence, $m = M$ and thus d_n is constant for all $n \in \mathbb{Z}^+$, i.e. $f(1), f(2), f(3), \ldots$ is an arithmetic sequence.

Therefore,

$$\sum_{j=1}^{100} f(j) = 1 + 3 + 5 + 7 + \cdots + 199 = \boxed{10000}.$$

10. Let $\mathbb{R}_{\geq 0}$ be the set of nonnegative real numbers. Consider a continuous function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ which satisfies
$$f(x^2) + f(y^2) = f\left(\frac{x^2y^2 - 2xy + 1}{x^2 + 2xy + y^2}\right)$$
for x, y positive real numbers with $xy > 1$. Given that $f(0) = 2019$ and $f(1) = \frac{2019}{2}$, compute $f(3)$.

Answer: 673

Solution: We begin by defining the functions $g(x) = f(x^2)$ so that the original equation becomes
$$g(x) + g(y) = g\left(\frac{xy - 1}{x + y}\right)$$
for all $x, y \in \mathbb{R}$. The term on the RHS suggests that we define a new function $h(\theta) = g(\cot\theta)$, which makes the equation become
$$h(\alpha) + h(\beta) = h(\alpha + \beta)$$
for all $\alpha, \beta \in \mathbb{R}$. This is the Cauchy functional equation, which has the family of solutions $h(\theta) = c\theta$ for some real constant c. Substituting f back in, we get $f(\cot^2\theta) = c\theta$. We solve $\cot^2\theta_0 = 0$ to get $\theta_0 = \frac{\pi}{2} + n\pi$ and $\cot^2\theta_1 = 1$ to get $\theta_1 = \frac{\pi}{4} + n\frac{\pi}{2}$. Note that because f is continuous, θ is restricted to the domain $[\frac{k\pi}{2}, \frac{(k+1)\pi}{2}]$ for some integer k. Hence, we must have $|\theta_0 - \theta_1| \leq \frac{\pi}{2}$ and thus, $|\frac{\pi}{4} + \frac{n\pi}{2}| \leq \frac{\pi}{2}$. So, $n = 0$ or -1. If $n = 0$, we see that $\theta_0 = \frac{\pi}{2}$ and $\theta_1 = \frac{\pi}{4}$ satisfy the conditions to give us $c = \frac{4038}{\pi}$. Thus, we see that $\theta \in [0, \frac{\pi}{2}]$, so solving $\cot^2\theta = 3$ in this interval gives us $\theta = \frac{\pi}{6}$, and therefore $f(3) = f(\cot^2\frac{\pi}{6}) = \frac{4038}{\pi} \cdot \frac{\pi}{6} = \boxed{673}$. (If $n = -1$, then $\theta_0 = -\frac{\pi}{2}$, $\theta_1 = -\frac{\pi}{4}$ and $c = \frac{4038}{\pi}$ with $\theta \in [-\frac{\pi}{2}, 0]$, giving $f(3) = 673$ as well.)