Comment: Version 1.1

JZ03 1. Suppose A and B are points in the plane lying on the parabola $y = x^2$, and the x-coordinates of A and B are -29 and 51, respectively. Let C be the point where line AB intersects the y-axis. What is the y-coordinate of C?

Answer: 1479

Solution: Since the coordinates of A and B are (x_1, x_1^2) and (x_2, x_2^2) for $x_1 = -29$ and $x_2 = 51$, it follows that the slope of line AB is

$$m = \frac{x_2^2 - x_1^2}{x_2 - x_1} = \frac{(x_2 + x_1)(x_2 - x_1)}{x_2 - x_1} = x_2 + x_1.$$

Since the line passes through $A = (x_1, x_1^2)$, the slope-point formula gives the formula of the line to be

$$\begin{aligned} y - x_1^2 &= (x_2 + x_1)(x - x_1) \\ y &= (x_2 + x_1)x - (x_2 + x_1)x_1 + x_1^2 \\ y &= (x_2 + x_1)x - x_1x_2 \end{aligned}$$

Thus the y-intercept of the line is $-x_1x_2 = -(-29)(51) = \boxed{1479}$.

HH01 2. Cindy has a collection of identical rectangular prisms. She stacks them, end to end, to form 1 longer rectangular prism. Suppose that joining 11 of them will form a rectangular prism with 3 times the surface area of an individual rectangular prism. How many will she need to join end to end to form a rectangular prism with 9 times the surface area?

Answer: 41

Solution 1: Let n be the number of boxes stacked together, and let m be the resulting multiplier on the surface area (i.e. the resulting box has m times the surface area of an individual box). Letting x, y be the lengths of the sides of the box and z be the height of the box, we may write

$$\begin{aligned} 2xy + 2x(nz) + 2y(nz) &= m(2xy + 2xz + 2yz) \\ (n - m)(x + y)z &= (m - 1)xy \\ \frac{n - m}{m - 1} &= \frac{xy}{(x + y)z}. \end{aligned}$$

Note that the right side is a constant because x, y, z are fixed. Furthermore, we are given $n = 11$ and $m = 3$, so plugging this in gives us $\frac{11-3}{3-1} = 4 = \frac{xy}{(x+y)z}$. Therefore, we must find n such that $\frac{n-9}{9-1} = 4$, which solves to $n = \boxed{41}$.

Solution 2: Note that after stacking 10 additional boxes, you gain 2 additional boxes worth of surface area. This ratio is constant, so if we need n additional boxes to get 8 additional boxes worth of surface area, we have the equation $\frac{n}{8} = \frac{10}{2}$. Solving yields $n = 40$, which means we need a total of $\boxed{41}$ boxes.

EX12 3. A lattice point is a point (a, b) on the Cartesian plane where a and b are integers. Compute the number of lattice points in the interior and on the boundary of the triangle with vertices at $(0, 0)$, $(0, 20)$, and $(18, 0)$.

Answer: 201

Solution: We first compute the number of lattice points on the segment from $(0, 20)$ to $(18, 0)$. The equation of the line connecting those two points is $10x + 9y = 180$, so lattice points are of the form $(18 - 9t, 10t)$ for some integer t. Counting, we find that the only lattice points on the line are $(0, 20)$, $(9, 10)$, and $(18, 0)$, so there are 3 points on the segment connecting $(0, 20)$ and $(18, 0)$.

Next, consider the rectangle with vertices at $(0, 0)$, $(0, 20)$, $(18, 20)$, and $(18, 0)$. The number of lattice points in or on the desired triangle is exactly half of the lattice points in the rectangle, if we add the points on the diagonal twice. The rectangle creates a 21×19 lattice grid, so the answer is $\dfrac{21 \cdot 19 + 3}{2} = \boxed{201}$.

KW04 4. Let $1 = a_1 < a_2 < a_3 < ... < a_k = n$ be the positive divisors of n in increasing order. If $n = a_3^3 - a_2^3$, what is n?

Answer: 56

Solution: We first consider the case where n is odd. Note that all of its factors, including a_2 and a_3, must be odd. However, because $n = a_3^3 - a_2^3$, n would then be the difference of two odd numbers, implying that n is even, a contradiction.

Therefore, n must be even, so $a_2 = 2$. Now suppose that a_3 is odd. Again, because $n = a_3^3 - 8$, n would then be the difference between an odd an and even number, implying that n is odd, another contradiction.

Therefore, a_3 must also be even. We can thus write $a_3 = 2k$ for some positive integer k. Note that k is also a factor of n and $k < 2k$, so we must have either $a_1 = k$ or $a_2 = k$. If $a_1 = k = 1$, then $a_3 = 2$, contradicting the fact that $a_2 < a_3$. Therefore, $a_2 = k = 2$, so $a_3 = 4$. Finally, we compute $n = 4^3 - 2^3 = \boxed{56}$.

KW16 5. A point (x_0, y_0) with integer coordinates is a primitive point of a circle if for some pair of integers (a, b), the line $ax + by = 1$ intersects the circle at (x_0, y_0). How many primitive points are there of the circle centered at $(2, -3)$ with radius 5?

Answer: 5

Solution: If there exists (a, b) such that $ax + by = 1$, then x and y must be coprime. We proceed by using the equation of a circle to find all of the integer points (x, y) where x and y are coprime.

We can describe the circle in the problem with the equation $(x-2)^2 + (y+3)^2 = 25$. Solving the equation in terms of y, we find that $y = -3 \pm \sqrt{-(x-2)^2 + 25}$. Hence, for (x, y) to be a lattice point, we need $-(x-2)^2 + 25$ to be a square. Enumerating over all possible values of x, we find that $-(x-2)^2 + 25$ is a square when $x = -3, -2, -1, 2, 5, 6, 7$. This gives us the following pairs for (x, y):

- $(-3, -3)$, which are not coprime
- $(-2, -6)$ and $(-2, 0)$, both of which are not coprime
- $(-1, -7)$ and $(-1, 1)$, both of which are coprime
- $(2, -8)$ and $(2, 2)$, both of which are not coprime
- $(5, -7)$ and $(5, 1)$, both of which are coprime
- $(6, -6)$ and $(6, 0)$, both of which are not coprime
- $(7, -3)$, which is coprime

In total, there are 5 (x, y) pairs such that x and y are coprime, so there are $\boxed{5}$ primitive points of the given circle.

LK02 6. Three distinct points are chosen uniformly at random from the vertices of a regular 2018-gon. What is the probability that the triangle formed by these points is a right triangle?

Answer: $\mathbf{\frac{3}{2017}}$

Solution: Consider the circle which circumscribes the regular 2018-gon. Note that all of the vertices of the regular 2018-gon lie on the circle. Thus, the 3 vertices chosen form a right triangle if and only if 2 of the vertices are diametrically opposite from each other. There are $\frac{2018}{2} = 1009$ ways for this to occur, and there are $2018 - 2 = 2016$ ways of choosing the additional third point of the right triangle.

On the other hand, there are $\binom{2018}{3}$ ways of choosing any 3 distinct vertices of the 2018-gon to form a triangle. Therefore, the probability of forming a right triangle is

$$\frac{1009 \cdot 2016}{\binom{2018}{3}} = \frac{1009 \cdot 2016}{\frac{2018 \cdot 2017 \cdot 2016}{3 \cdot 2}} = \boxed{\frac{3}{2017}}.$$

KW01 7. Consider any 5 points placed on the surface of a cube of side length 2 centered at the origin. Let m_x be the minimum distance between the x coordinates of any of the 5 points, m_y be the minimum distance between y coordinates, and m_z be the minimum distance between z coordinates. What is the maximum value of $m_x + m_y + m_z$?

Answer: $\mathbf{\frac{3}{2}}$

Solution: Applying the pigeonhole principle to each axis, we see that the maximum values of m_x, m_y, m_z are all $\frac{1}{2}$. Furthermore, in this case the coordinates of each of the points must lie in the set $\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$. It remains to construct values for the 5 points that will work. Note that a point is on the surface of the cube if at least one of the coordinates of a point has absolute value 1. We have 6 coordinates with absolute value 1, and only 5 points, so we can easily choose them to lie on the surface of the cube and satisfy the desired properties. For example, the points

$$\left(-1, \frac{1}{2}, 0\right), \left(-\frac{1}{2}, 1, -1\right), \left(0, -1, -\frac{1}{2}\right), \left(\frac{1}{2}, 0, 1\right), \left(1, -\frac{1}{2}, \frac{1}{2}\right)$$

satisfy the given conditions. Therefore, the maximum value is $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \boxed{\frac{3}{2}}$.

EY04 8. Eddy has two blank cubes A and B and a marker. Eddy is allowed to draw a total of 36 dots on cubes A and B to turn them into dice, where each side has an equal probability of appearing, and each side has at least one dot on it. Eddy then rolls dice A twice and dice B once and computes the product of the three numbers. Given that Eddy draws dots on the two dice to maximize his expected product, what is his expected product?

Answer: 32

Solution: Suppose the faces of dice A are labeled as $a_1, ..., a_6$ and the faces of dice B are labeled as $b_1, ..., b_6$. We are given that $a_1 + \cdots + a_6 + b_1 + \cdots + b_6 = 36$. We can then write the expected product as

$$\frac{1}{6^3}(a_1 + \cdots + a_6)^2(b_1 + \cdots + b_6).$$

If we let $a = a_1 + \cdots + a_6$ and $b = b_1 + \cdots + b_6$, then we can reduce this to maximizing $\frac{a^2 b}{6^3}$ under the constraint $a + b = 36$.

Note that it suffices to maximize $a^2 b$, which we can do using AM-GM as follows:

$$\frac{a+b}{3} = \frac{a/2 + a/2 + b}{3} \geq \sqrt[3]{\frac{a^2 b}{4}}$$

where equality (and thus the maximum) holds when $a/2 = b$. Since $a + b = 36$, equality occurs when $a = 24$ and $b = 12$, giving us a maximum expected value of $\frac{24^2 \cdot 12}{6^3} = 4^2 \cdot 2 = \boxed{32}$.

EX03 9. Let $ABCD$ be a square. Point E is chosen inside the square such that $AE = 6$. Point F is chosen outside the square such that $BE = BF = 2\sqrt{5}$, $\angle ABF = \angle CBE$, and $AEBF$ is cyclic. Compute the area of $ABCD$.

Answer: 32

Solution: Since $\angle ABF = \angle CBE$, we have $\angle EBF = \angle CBA = 90°$. Moreover, since $BE = BF = 2\sqrt{5}$, $\triangle EBF$ is a 45-45-90 triangle, so $EF = 2\sqrt{5} \cdot \sqrt{2} = 2\sqrt{10}$. Because $AEBF$ is cyclic, $\angle EBF + \angle EAF = 180°$, so $\angle EAF = 90°$. By the Pythagorean Theorem, we find that $AF = \sqrt{(2\sqrt{10})^2 - 6^2} = 2$. We can then apply Ptolemy's Theorem on cyclic quadrilateral $AEBF$ to get $6(2\sqrt{5}) + 2(2\sqrt{5}) = AB(2\sqrt{10})$. Solving, we get $AB = 4\sqrt{2}$, so the area of $ABCD$ is $(4\sqrt{2})^2 = \boxed{32}$.

HH20 10. Find the total number of sets of nonnegative integers (w, x, y, z) where $w \leq x \leq y \leq z$ such that $5w + 3x + y + z = 100$.

Answer: 2156

Solution: The constraint that $w \leq x \leq y \leq z$ is pesky, so we attempt to remove it. Let $a = w$, $b = x - w$, $c = y - x$, and $d = z - y$. Observe that a, b, c, d are all nonnegative integers, since w, x, y, z are nonnegative. Then the number of solutions to the given equation is equivalent to the number of solutions to the equation

$$5a + 3(a + b) + (a + b + c) + (a + b + c + d) = 10a + 5b + 2c + d = 100.$$

Next, we compute the total number of combinations by considering the sum in increments of 10, where there are a total of $100/10 = 10$ increments to consider. There are 3 possible cases:

(a) The only increments of 10 are 10, $5 + 5$, $2 + 2 + 2 + 2 + 2$, and $1 + 1 + ... + 1 = 1 \cdot 10$. The number of solutions is equivalent to computing the number of ways of placing 10 balls in 4 urns, or of placing 3 dividers in 10 increments. Hence, there are $\binom{13}{3} = 286$ possible solutions.

(b) In addition to the above 4 increments, there is 1 increment of 10 consisting of some combination of 5s, 2s, and 1s. In total, there are 7 ways of achieving this: 2, $2 + 2$, $2 + 2 + 2$, $2 + 2 + 2 + 2$, 5, $5 + 2$, and $5 + 2 + 2$ (with each sum padded with 1s to equal 10). The number of ways of placing the 4 even increments of 10, $5 \cdot 2$, $2 \cdot 5$, and $1 \cdot 10$ is equivalent to the number of ways of placing 3 dividers in 9 increments. Hence, there are $\binom{12}{3} \cdot 7 = 220 \cdot 7 = 1540$ possible solutions.

(c) In addition to the above 4 increments, there are 2 increments of 10 consisting of some combination of 5s, 2s, and 1s. In total, there are 2 ways of achieving this: $5 + 2 + 2 + 2$ and $5 + 2 + 2 + 2 + 2$ (with each sum padded with 1s to equal 10). The number of ways of placing the 4 even increments is equivalent to the number of ways of placing 3 dividers in 8 increments. Hence, there are $\binom{11}{3} \cdot 2 = 165 \cdot 2 = 330$ possible solutions.

Note that there are no possible cases, as adding additional 5s or 2s would result in additional increments of $5 + 5$ or $2 + 2 + 2 + 2 + 2$.

In total, there are $286 + 1540 + 330 = \boxed{2156}$ total solutions to the original equation.

LK05 11. Let $f(k)$ be a function defined by the following rules:

(a) $f(k)$ is multiplicative. In other words, if $\gcd(a, b) = 1$, then $f(ab) = f(a) \cdot f(b)$,

(b) $f(p^k) = k$ for primes $p = 2, 3$ and all $k > 0$,

(c) $f(p^k) = 0$ for primes $p > 3$ and all $k > 0$, and

(d) $f(1) = 1$.

For example, $f(12) = 2$ and $f(160) = 0$. Evaluate

$$\sum_{k=1}^{\infty} \frac{f(k)}{k}.$$

Answer: $\mathbf{\frac{21}{4}}$

Solution: Note that when k is not divisible by only 2 and 3, $f(k) = 0$. Therefore, we are only concerned with numbers of the form $k = 2^m \cdot 3^n$, where m and n are non-negative integers. Furthermore, because $f(k)$ is multiplicative, if $k = 2^m \cdot 3^n$, then

$$\frac{f(k)}{k} = \frac{f(2^m \cdot 3^n)}{2^m \cdot 3^n} = \frac{f(2^m)}{2^m} \cdot \frac{f(3^n)}{3^n} = \frac{m}{2^m} \cdot \frac{n}{2^n}.$$

This allows us to rewrite our sum as the following product of sums

$$\sum_{k=1}^{\infty} \frac{f(k)}{k} = \left(1 + \sum_{m=1}^{\infty} \frac{m}{2^m}\right)\left(1 + \sum_{n=1}^{\infty} \frac{n}{3^n}\right).$$

Note that each term $\frac{f(2^m \cdot 3^n)}{2^m \cdot 3^n}$ appears when we multiply $\frac{m}{2^m}$ from the left sum with $\frac{n}{2^n}$ from the right sum. If $m = 0$, then the term $\frac{f(3^n)}{3^n} = \frac{n}{3^n}$ appears when we multiply the 1 in the left sum with the $\frac{n}{3^n}$ term in the right sum. A similar case happens when $n = 0$.

Now let $S = \sum_{m=1}^{\infty} \frac{m}{2^m}$. We can calculuate

$$\begin{aligned} S - \frac{S}{2} &= \sum_{m=1}^{\infty} \frac{m}{2^m} - \sum_{m=1}^{\infty} \frac{m}{2^{m+1}} \\ &= \frac{1}{2} + \sum_{m=2}^{\infty} \frac{m}{2^m} - \sum_{m=2}^{\infty} \frac{m-1}{2^m} \\ &= \frac{1}{2} + \sum_{m=2}^{\infty} \frac{1}{2^m} \\ &= \sum_{m=1}^{\infty} \frac{1}{2^m} \\ &= 1 \end{aligned}$$

Therefore, $S = 2$. Using a similar approach, we find that $\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{3}{4}$. Plugging these into our product of sums, our original sum is thus $(1+2)\left(1 + \frac{3}{4}\right) = \boxed{\frac{21}{4}}$.

LK12 12. Consider all increasing arithmetic progressions of the form $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ such that $a, b, c \in \mathbb{N}$ and $\gcd(a, b, c) = 1$. Find the sum of all possible values of $\frac{1}{a}$.

Answer: $\mathbf{2\ln 2 - 1}$

Solution: Note: The solution below is incorrect, as it misses many triples (a, b, c) such as $(6, 4, 3)$. During the grading period, we were unable to determine the correct solution, and so gave all teams credit for this problem.

Since $\frac{1}{a}$ is the initial term, the other subsequent terms are $\frac{1+r}{a}$ and $\frac{1+2r}{a}$ for some common ratio r. To ensure that the subsequent 2 terms in the arithmetic progression have a numerator of 1, we must have $1 + r \mid a$ and $1 + 2r \mid a$. Therefore, $a = k(1 + r)(1 + 2r)$. However, we must have $k = 1$, otherwise $\gcd(a, b, c) \geq k > 1$. Thus, $a = (1+r)(1+2r)$. Next, note that r must be an integer, otherwise we can rearrange the above equation to form $\frac{1}{c} = \frac{1+2r}{a} = \frac{1}{1+r}$ which would not be a valid fraction. Finally, we verify that every $r \geq 1$ indeed gives us an increasing arithmetic progression $\frac{1}{(1+r)(1+2r)}, \frac{1}{1+2r}, \frac{1}{1+r}$ with common ratio $\frac{r}{(1+r)(1+2r)}$. Therefore, it remains to compute the sum

$$\begin{aligned}
\sum_{r=1}^{\infty} \frac{1}{(1+r)(1+2r)} &= \sum_{r=1}^{\infty} \frac{2}{1+2r} - \frac{1}{1+r} \\
&= \sum_{r=1}^{\infty} \frac{2}{1+2r} - \frac{2}{2+2r} \\
&= 2\sum_{r=1}^{\infty} \frac{1}{1+2r} - \frac{1}{2+2r} \\
&= 2\left(\frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right)
\end{aligned}$$

The inner sum looks very similar to the Taylor series expansion of

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Therefore, our desired sum is $2\left(\ln 2 - \frac{1}{2}\right) = \boxed{2\ln 2 - 1}$.

EX19 13. In $\triangle ABC$, let D, E, and F be the feet of the altitudes drawn from A, B, and C respectively. Let P and Q be points on line EF such that BP is perpendicular to EF and CQ is perpendicular to EF. If $PQ = 2018$ and $DE = DF + 4$, find DE.

Answer: 1011

Solution: Note that $\triangle DEF$ is the orthic triangle of $\triangle ABC$, so A, B, and C are excenters of $\triangle DEF$. Let ω_1 be the excircle centered at B and ω_2 be the excircle centered at C. Since $BP \perp EF$ and $CQ \perp EF$, PQ is an external tangent to ω_1 and ω_2. Furthermore, lines DE and DF are internal tangents to ω_1 and ω_2. Let line DF touch ω_1 at X and ω_2 at Y. Let P' and Q' be on ω_1 and ω_2 respectively such that $P'Q'$ is the other external tangent to ω_1 and ω_2. Let line DF intersect $P'Q'$ at E'.

Since tangents from a point to a circle are equal in length, we have $FP = FX$, $FQ = FY$, $E'P' = E'X$, and $E'Y = E'Q'$. By symmetry of external tangents, we also have $PQ = P'Q'$. Then

$$\begin{aligned}
2PQ &= PQ + P'Q' \\
&= (FP + FQ) + (E'P' + E'Q') \\
&= (FX + FY) + (E'X + E'Y) \\
&= (FX + E'X) + (FY + E'Y') \\
&= E'F + E'F \\
&= 2E'F
\end{aligned}$$

so $PQ = E'F$. But by symmetry of internal and external tangents, we see that E and E' are reflections of each other across BC, and since D lies on BC, we have $DE = DE'$. Then $PQ = E'F = DE' + DF = DE + DF$. It follows that $DE + (DE - 4) = 2018$ and so $DE = \boxed{1011}$.

NM04 14. Let A and B be two points chosen independently and uniformly at random inside the unit circle centered at O. Compute the expected area of $\triangle ABO$.

Answer: $\frac{4}{9\pi}$

Solution: Recall that we can compute the area of $\triangle ABC$ using the formula $A = \frac{1}{2}ab\sin C$. Furthermore, for independent random variables X and Y, we have $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

By assumption, points A and B are chosen independently of each other, so the lengths of OA and OB are also independent. Furthermore, if we let θ be the smaller angle between OA and OB, then θ is also independent of the two side lengths. Therefore, we have

$$\mathbb{E}[A] = \mathbb{E}\left[\frac{1}{2} \cdot OA \cdot OB \cdot \sin\theta\right] = \frac{1}{2} \cdot \mathbb{E}[OA] \cdot \mathbb{E}[OB] \cdot \mathbb{E}[\sin\theta].$$

Now for a point P chosen uniformly at random in a circle, the probability that $OP = r$ is proportional to r. We can show this by recalling that the circle of radius r has circumference $2\pi r$, so if $r_1 = kr_2$, then the set of points of distance r_1 from the center of the circle is k times larger than the set of points of distance r_2 from the center of the circle. Therefore, the probability density function of the length of OP is $f(r) = cr$ when $r \in [0, 1]$ and 0 otherwise, for some constant c. To find c, we solve $\int_0^1 cr\,dr = 1$, which gives us $c = 2$. Therefore, the expected length of OP is

$$\mathbb{E}[OP] = \int_0^1 r \cdot 2r\,dr = \frac{2}{3}.$$

On the other hand, θ is distributed uniformly at random from the interval $[0, \pi]$ by symmetry. Therefore, the expected value of $\sin\theta$ is

$$\mathbb{E}[\sin\theta] = \frac{1}{\pi}\int_0^\pi \sin\theta\,d\theta = \frac{2}{\pi}.$$

Plugging in these values, we get $\mathbb{E}[A] = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{\pi} = \boxed{\frac{4}{9\pi}}$.

HH19 15. Suppose that a, b, c, d are positive integers satisfying

$$\begin{aligned} 25ab + 25ac + b^2 &= 14bc \\ 4bc + 4bd + 9c^2 &= 31cd \\ 9cd + 9ca + 25d^2 &= 95da \\ 5da + 5db + 20a^2 &= 16ab \end{aligned}$$

Compute $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$.

Answer: $\frac{161}{30}$

Solution: We begin by moving all variables to the left hand side of the equations.

$$\begin{aligned} 25 \cdot \frac{a}{c} + 25 \cdot \frac{a}{b} + \frac{b}{c} &= 14 \\ 4 \cdot \frac{b}{d} + 4 \cdot \frac{b}{c} + 9 \cdot \frac{c}{d} &= 31 \\ 9 \cdot \frac{c}{a} + 9 \cdot \frac{c}{d} + 25 \cdot \frac{d}{a} &= 95 \\ 5 \cdot \frac{d}{b} + 5 \cdot \frac{d}{a} + 20 \cdot \frac{a}{b} &= 16 \end{aligned}$$

Next, we apply Simon's Favorite Factoring Trick

$$25 \cdot \frac{a}{c} + 25 \cdot \frac{a}{b} + \frac{b}{c} + \frac{b}{b} = 15$$
$$4 \cdot \frac{b}{d} + 4 \cdot \frac{b}{c} + 9 \cdot \frac{c}{d} + 9 \cdot \frac{c}{c} = 40$$
$$9 \cdot \frac{c}{a} + 9 \cdot \frac{c}{d} + 25 \cdot \frac{d}{a} + 25 \cdot \frac{d}{d} = 120$$
$$5 \cdot \frac{d}{b} + 5 \cdot \frac{d}{a} + 20 \cdot \frac{a}{b} + 20 \cdot \frac{a}{a} = 36$$

and factor to get

$$(25a + b)\left(\frac{1}{c} + \frac{1}{b}\right) = 15$$
$$(4b + 9c)\left(\frac{1}{d} + \frac{1}{c}\right) = 40$$
$$(9c + 25d)\left(\frac{1}{a} + \frac{1}{d}\right) = 120$$
$$(5d + 20a)\left(\frac{1}{b} + \frac{1}{a}\right) = 36$$

Multiplying all of the equations together, and grouping the same pairs of variables gives us

$$\left[(25a + b)\left(\frac{1}{a} + \frac{1}{b}\right)\right] \cdot \left[(4b + 9c)\left(\frac{1}{b} + \frac{1}{c}\right)\right]$$
$$\cdot \left[(9c + 25d)\left(\frac{1}{c} + \frac{1}{d}\right)\right] \cdot \left[(5d + 20a)\left(\frac{1}{d} + \frac{1}{a}\right)\right] = 15 \cdot 40 \cdot 120 \cdot 36$$
$$\left[25 + 1 + 25 \cdot \frac{a}{b} + \frac{b}{a}\right] \cdot \left[4 + 9 + 4 \cdot \frac{b}{c} + 9 \cdot \frac{c}{b}\right]$$
$$\cdot \left[9 + 25 + 9 \cdot \frac{c}{d} + 25 \cdot \frac{d}{c}\right] \cdot \left[5 + 20 + 5 \cdot \frac{d}{a} + 20 \cdot \frac{a}{d}\right] = 15 \cdot 40 \cdot 120 \cdot 36$$

By AM-GM, we have $\frac{25 \cdot \frac{a}{b} + \frac{b}{a}}{2} \geq \sqrt{25 \cdot \frac{a}{b} \cdot \frac{b}{a}} = 5$, so $25 \cdot \frac{a}{b} + \frac{b}{a} \geq 10$. In a similar fashion, we also find that $4 \cdot \frac{b}{c} + 9 \cdot \frac{c}{b} \geq 12$, $9 \cdot \frac{c}{d} + 25 \cdot \frac{d}{c} \geq 30$, and $5 \cdot \frac{d}{a} + 20 \cdot \frac{a}{d} \geq 20$. This gives us

$$\left[25 + 1 + 25 \cdot \frac{a}{b} + \frac{b}{a}\right] \cdot \left[4 + 9 + 4 \cdot \frac{b}{c} + 9 \cdot \frac{c}{b}\right]$$
$$\cdot \left[9 + 25 + 9 \cdot \frac{c}{d} + 25 \cdot \frac{d}{c}\right] \cdot \left[5 + 20 + 5 \cdot \frac{d}{a} + 20 \cdot \frac{a}{d}\right]$$
$$\geq (26 + 10)(13 + 12)(34 + 30)(25 + 20)$$
$$= 36 \cdot 25 \cdot 64 \cdot 45$$
$$= 15 \cdot 40 \cdot 120 \cdot 36$$

Since equality holds, we must have

$$25 \cdot \frac{a}{b} + \frac{b}{a} = 10$$
$$4 \cdot \frac{b}{c} + 9 \cdot \frac{c}{b} = 12$$
$$9 \cdot \frac{c}{d} + 25 \cdot \frac{d}{c} = 30$$
$$5 \cdot \frac{d}{a} + 20 \cdot \frac{a}{d} = 20$$

Solving these equations individually, we get $\frac{a}{b} = \frac{1}{5}$, $\frac{b}{c} = \frac{3}{2}$, $\frac{c}{d} = \frac{5}{3}$, and $\frac{d}{a} = 2$. Therefore, our desired answer is $\frac{1}{5} + \frac{3}{2} + \frac{5}{3} + 2 = \boxed{\frac{161}{30}}$.

KW30 1. Prove that if 7 divides $a^2 + b^2 + 1$, then 7 does not divide $a + b$.

Answer: See proof

Solution: We first note that the only possible squares mod 7 are $0, 1, 2, 4$. Now, if $a^2+b^2+1 \equiv 0 \pmod 7$, then we must have $a^2 + b^2 \equiv 6 \pmod 7$. Therefore, we must have either $a^2 \equiv 2 \pmod 7$ and $b^2 \equiv 4 \pmod 7$ or vice versa. WLOG suppose that $a^2 \equiv 2 \pmod 7$ and $b^2 \equiv 4 \pmod 7$. Taking the square root of both sides yields $a \equiv \pm 3 \pmod 7$ and $b \equiv \pm 2 \pmod 7$. However, we never have $a+b \equiv 0 \pmod 7$ given these values of a and b, so 7 does not divide $a + b$.

EY03 2. Consider a game played on the integers in the closed interval $[1, n]$. The game begins with some tokens placed in $[1, n]$. At each turn, tokens are added or removed from $[1, n]$ using the following rule: For each integer $k \in [1, n]$, if exactly one of $k - 1$ and $k + 1$ has a token, place a token at k for the next turn, otherwise leave k blank for the next turn.

We call a position static if no changes to the interval occur after one turn. For instance, the trivial position with no tokens is static because no tokens are added or removed after a turn (because there are no tokens). Find all non-trivial static positions.

Answer: See proof

Solution: We claim that a non-trivial static position exists if and only if $n \equiv 2 \pmod 3$. To show this, consider a token at position k. For k to not change, it must have exactly one neighbor. WLOG, let it be at $k + 1$ so that $k - 1$ is empty. For $k - 1$ to not change, either $k - 1 = 0$ or $k - 2$ has a token so that $k - 1$ has two neighboring tokens. Then $k - 3$ must also have a token. However, the neighbors of $k - 3$ look exactly like k, so we must have $k - 3i = 1$ for some $i \geq 0$. We can use a similar argument to show that $k + 1 + 3j = n$ for some $j \geq 0$ as well. The interval must therefore be of length $k + 1 + 3j - (k - 3i) + 1 = 3(i + j) + 2$, as desired. Note that we can construct a non-trivial static position for all $n \equiv 2 \pmod 3$ by choosing appropriate values of i and j. For example, the static position for $n = 5$ will look like `XX_XX`, where `X` denotes a token and `_` denotes a blank space.

KW24 3. Show that if A is a shape in the Cartesian coordinate plane with area greater than 1, then there are distinct points $(a, b), (c, d)$ in A where $a - c = 2x + 5y$ and $b - d = x + 3y$ where x, y are integers.

Answer: See proof

Solution: In vector notation, $\begin{pmatrix} a \\ b \end{pmatrix}$ denotes the vector (a, b). We note that the equations mean that $\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} c \\ d \end{pmatrix} = x \begin{pmatrix} 2 \\ 1 \end{pmatrix} + y \begin{pmatrix} 5 \\ 3 \end{pmatrix}$. Now, notice that we can write $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 5 \\ 3 \end{pmatrix} - 5 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Hence, if we can write $\begin{pmatrix} a - c \\ b - d \end{pmatrix}$ as $\begin{pmatrix} m \\ n \end{pmatrix}$ where m, n are integers, then we are done.

Now place A on top of the integer grid. We cut A into pieces using the integer axes, and translate all of the pieces onto the square bounded by $(0, 0), (0, 1), (1, 0), (1, 1)$. Note that because we cut A using the integer axes, all translations are shifts using integer vectors. Furthermore, shifting will not change the area contained in the grid. Because there is an overlap between pieces in this square, there exist points (a, b) in one piece and (c, d) in another piece that differ by an integer vector since they were shifted using integer vectors. However, the area of A is greater than 1, the area of the unit square, and by the Pigeonhole Principle, there must exist two pieces which overlap. Thus, we are done.

EY21 4. Let F_k denote the series of Fibonacci numbers shifted back by one index, so that $F_0 = 1$, $F_1 = 1$, and $F_{k+1} = F_k + F_{k-1}$. It is known that for any fixed $n \geq 1$ there exist real constants

b_n, c_n such that the following recurrence holds for all $k \geq 1$:

$$F_{n\cdot(k+1)} = b_n \cdot F_{n\cdot k} + c_n \cdot F_{n\cdot(k-1)}.$$

Prove that $|c_n| = 1$ for all $n \geq 1$.

Answer: See proof

Solution: We begin with the following observation. We can rewrite the recurrence $F_n = F_{n-1} + F_{n-2}$ as $F_n = F_1 \cdot F_{n-1} + F_0 \cdot F_{n-2}$. Substituting in $F_{n-1} = F_{n-2} + F_{n-3}$ gives us

$$F_n = F_1 \cdot (F_{n-2} + F_{n-3}) + F_0 \cdot F_{n-2} = F_2 \cdot F_{n-2} + F_1 \cdot F_{n-3}.$$

Continuing along this manner, we see that for any $0 < k < n$ we can in fact write

$$F_n = F_k \cdot F_{n-k} + F_{k-1} \cdot F_{n-k-1}.$$

Returning to the original problem, we can therefore write

$$\begin{aligned} F_{n\cdot(k+1)} &= F_{2n-1} \cdot F_{n\cdot(k-1)+1} + F_{2n-2} \cdot F_{n\cdot(k-1)} \\ F_{n\cdot k} &= F_{n-1} \cdot F_{n\cdot(k-1)+1} + F_{n-2} \cdot F_{n\cdot(k-1)}. \end{aligned}$$

Because we know that

$$F_{n\cdot(k+1)} = b_n \cdot F_{n\cdot k} + c_n \cdot F_{n\cdot(k-1)}$$

we therefore have the system of equations

$$\begin{aligned} F_{2n-1} &= b_n \cdot F_{n-1} \\ F_{2n-2} &= b_n \cdot F_{n-2} + c_n \end{aligned}$$

Solving for b_n, we find

$$b_n = \frac{F_{2n-1}}{F_{n-1}} = \frac{F_{n-1} \cdot F_n + F_{n-2} \cdot F_{n-1}}{F_{n-1}} = F_n + F_{n-2}.$$

Plugging this into the second equation, we have

$$\begin{aligned} c_n &= F_{2n-2} - F_{n-2}(F_n + F_{n-2}) \\ &= F_{n-1} \cdot F_{n-1} + F_{n-2} \cdot F_{n-2} - F_{n-2} \cdot F_n - F_{n-2} \cdot F_{n-2} \\ &= F_{n-1} \cdot F_{n-1} - F_n \cdot F_{n-2}. \end{aligned}$$

We prove that this last equation is ± 1 by induction on n. For the base case, $n = 2$, we have $F_1 \cdot F_1 - F_2 \cdot F_0 = 1 - 2 = -1$. For the inductive step, suppose that $F_{n-2} \cdot F_{n-2} - F_{n-1} \cdot F_{n-3} = \pm 1$. Then we have

$$\begin{aligned} F_{n-1} \cdot F_{n-1} - F_n \cdot F_{n-2} &= F_{n-1} \cdot F_{n-1} - F_{n-1} \cdot F_{n-2} - F_{n-2} \cdot F_{n-2} \\ &= F_{n-1} \cdot F_{n-3} - F_{n-2} \cdot F_{n-2} \\ &= \mp 1 \end{aligned}$$

completing the inductive step and the proof.

KW43 5. Let $ABCD$ be a quadrilateral with sides AB, BC, CD, DA and diagonals AC, BD. Suppose that all sides of the quadrilateral have length greater than 1, and that the difference between any side and diagonal is less than 1. Prove that the following inequality holds:

$$(AB + BC + CD + DA + AC + BD)^2 > 2|AC^3 - BC^3| + 2|BD^3 - AD^3| - (AB + CD)^3$$

Answer: See proof

Solution: Let $P = (AB+BC+CD+DA+AC+BD)^2$, $S = 2|AC^3-BC^3|+2|BD^3-AD^3|$, and $Q = (AB+CD)^3$, so we can rewrite the expression as $P+Q > S$.

Looking at the expression, we expect a fairly algebraic argument boiling down to the triangle inequality. Thus, we try to reduce the expression to linear factors.

First, we reduce the expression to quadratic factors. Note that since the side lengths are all greater than 1, we have $Q = (AB+CD)^3 > (AB+CD)^2$. Let $Q' = (AB+CD)^2$. It now suffices to show that $P + Q' > S$.

Now becase the difference between any side and diagonal is less than 1, we have

$$\begin{aligned}
S &= 2|AC^3 - BC^3| + 2|BD^3 - AD^3| \\
&\leq 2\left(\frac{AC^3 - BC^3}{AC - BC}\right) + 2\left(\frac{BD^3 - AD^3}{BD - AD}\right) \\
&= 2(AC^2 + BC^2 + AC \cdot BC) + 2(AD^2 + BD^2 + AD \cdot BD) \\
&< 2(AC^2 + BC^2 + AD^2 + BD^2) + 2(AC \cdot BC + AC \cdot AD + BC \cdot BD + AD \cdot BD)
\end{aligned}$$

Let the final expression above be S'. It now suffices to show that $P + Q' > S'$. Rearranging, we find this is equivalent to showing that

$$\begin{aligned}
&(AB + BC + CD + DA + AC + BD)^2 + (AB + CD)^2 \\
&\quad - 2(AC \cdot BC + AC \cdot AD + BC \cdot BD + AD \cdot BD) > 2(AC^2 + BC^2 + AD^2 + BD^2)
\end{aligned}$$

We can then rewrite the left hand side as

$$\begin{aligned}
(AB + BC + CD + DA)^2 + (AC + BD)^2 &+ 2(AB + BC + CD + DA)(AC + BD) \\
&+ (AB + CD)^2 - 2(AC \cdot BC + AC \cdot AD + BC \cdot BD + AD \cdot BD) \\
&= (AB + BC + CD + DA)^2 + (AC + BD + AB + CD)^2
\end{aligned}$$

Incorporating this into the left hand side of the inequality above and rearranging, we have

$$\begin{aligned}
(AB + BC + CD + DA)^2 &+ (AB + AC + BD + CD)^2 \\
&- 2(AC^2 + BC^2 + AD^2 + BD^2) > 0
\end{aligned}$$

Multiplying both sides by 2 and expanding gives us

$$\begin{aligned}
&2(AB + BC + CD + DA)^2 - 4(BD^2 + AC^2) \\
&\quad + 2(AB + AC + CD + BD)^2 - 4(BC^2 + AD^2) \\
&= (AB + BC + CD + DA)^2 - (2BD)^2 + (AB + BC + CD + DA)^2 - (2AC)^2 \\
&\quad + (AB + AC + CD + BD)^2 - (2BC)^2 \\
&\quad + (AB + AC + CD + BD)^2 - (2AD)^2 > 0
\end{aligned}$$

Note that $(AB+BC+CD+DA)^2 - (2BD)^2 = (AB+BC+CD+DA-2BD)(AB+BC+CD+DA+2BD)$. By the triangle inequality, $BC+CD > BD$ and $DA+AB > BD$, hence $AB+BC+CD+DA-2BD > 0$ and $(AB+BC+CD+DA)^2 - (2BD)^2 > 0$. Repeating this for the other 3 pairs of differences in the inequality, we find that each pair is greater than 0. Hence, the inequality does hold. All of our steps are reversible, so the original inequality also holds, as desired.