1. Consider a semi-circle with diameter AB. Let points C and D be on diameter AB such that CD forms the base of a square inscribed in the semicircle. Given that $CD = 2$, compute the length of AB.

   Answer: $2\sqrt{5}$

   Solution: Note that the center of the semi-circle lies on the center of one of the sides of the square. If we draw a line from the center to an opposite corner of the square, we form a right triangle whose side lengths are 1 and 2 and whose hypotenuse is the radius of the semicircle. We can therefore use the Pythagorean Theorem to compute $r = \sqrt{1^2 + 2^2} = \sqrt{5}$. The radius is half the length of AB so therefore $AB = \boxed{2\sqrt{5}}$.

2. Let $ABCD$ be a trapezoid with AB parallel to CD and perpendicular to BC. Let M be a point on BC such that $\angle AMB = \angle DMC$. If $AB = 3$, $BC = 24$, and $CD = 4$, what is the value of $AM + MD$?

   Answer: 25

   Solution: Let A' be the reflection of A by BC. We have $\angle A'MB = \angle AMB = \angle DMC$. Hence, A', M, and D are collinear. Let C' be the intersection of the line parallel to BC passing through A' and the extension of DC. We have $\angle A'C'D = 90°$, $A'C' = BC = 24$, and $C'D = C'C + CD = A'B + CD = AB + CD = 3 + 4 = 7$. Therefore, $AM + MD = A'M + MD = A'D = \sqrt{A'C'^2 + C'D^2} = \sqrt{24^2 + 7^2} = \boxed{25}$ by Pythagorean Theorem.

3. Let ABC be a triangle and D be a point such that A and D are on opposite sides of BC. Give that $\angle ACD = 75°$, $AC = 2$, $BD = \sqrt{6}$, and AD is an angle bisector of both $\triangle ABC$ and $\triangle BCD$, find the area of quadrilateral $ABDC$.

   Answer: $3 + \sqrt{3}$

   Solution 1: Since AD is an angle bisector of both ABC and BCD, $\angle BAD = \angle CAD$ and $\angle BDA = \angle CDA$. Then by angle-side-angle congruence, $\triangle ABD \cong \triangle ACD$, and $CD = BD = \sqrt{6}$. Since $\angle ACD = 75°$, we can use the area formula

$$[ACD] = \frac{1}{2}AC \cdot CD \cdot \sin 75° = \frac{1}{2} \cdot 2 \cdot \sqrt{6} \cdot \frac{\sqrt{2} + \sqrt{6}}{4} = \frac{\sqrt{3} + 3}{2}.$$

   Because $[ABD] = [ACD]$ we have $[ABCD] = 2[ACD] = \boxed{3 + \sqrt{3}}$.

   Solution 2: We begin as in the original solution by noticing $\triangle ABD \cong \triangle ACD$. This implies that $ABCD$ is a kite, which means that $AD \perp BC$. Let E be the intersection of AD and BC. Note that then $\triangle CAE$ is a $30-60-90$ right triangle, while $\triangle CDE$ is a $45-45-90$ right triangle. This gives us $AD = 1 + \sqrt{3}$ and $BC = 2\sqrt{3}$, so the area of kite $ABCD$ is $\frac{1}{2}AD \cdot BC = \boxed{3 + \sqrt{3}}$.

4. Let $a_1, a_2, ..., a_{12}$ be the vertices of a regular dodecagon D_1 (12-gon). The four vertices a_1, a_4, a_7, a_{10} form a square, as do the four vertices a_2, a_5, a_8, a_{11} and a_3, a_6, a_9, a_{12}. Let D_2 be the polygon formed by the intersection of these three squares. If we let $[A]$ denotes the area of polygon A, compute $\frac{[D_2]}{[D_1]}$.

   Answer: $4 - 2\sqrt{3}$

   Solution: By symmetry, D_2 is also a regular dodecagon. Therefore, to find the ratio of areas, we need only find the ratio of side lengths between the two dodecagons.

   We begin by labeling relevant points and lines. Let X be the intersection of $a_{12}a_3$ and a_2a_5; Y be the intersection of $a_{12}a_3$ and a_1a_4; and Z be the intersection of a_1a_4 and a_2a_5. Then YZ is a side length of D_2, so we must find the ratio YZ/a_2a_3. Note that a_1a_4 is parallel to a_2a_3, so XY is also parallel to a_2a_3, which implies that $\triangle a_2a_3X \sim \triangle ZYX$. Thus, $\frac{YZ}{a_2a_3} = \frac{XZ}{a_2X}$.

Let x denote the length of a_2a_3. Since D_1 is a 12-gon, each angle is $\frac{10\cdot 180}{12} = 150$ degrees. We know that $\angle a_{11}a_2a_5$ is the right angle of a square, and by symmetry $\angle a_{11}a_2a_1 = \angle a_5a_2a_3$, so $\angle Xa_2a_3 = \frac{150-90}{2} = 30$. Again, by symmetry we also have $\angle Xa_3a_2 = 30$. Using $30-60-90$ right triangles, we see that $a_2X = x/\sqrt{3}$. On the other hand, since $\angle Za_1a_2 = 30$ by symmetry and $\angle a_1Za_2 = \angle YZX = 30$ by similar triangles, we see that $\triangle a_1Za_2$ is isoceles, and therefore $a_2Z = x$. Since $XZ = a_2Z - a_2X$, we have $XZ = x - x/\sqrt{3}$. Therefore, $\frac{XZ}{a_2X} = \frac{x-x/\sqrt{3}}{x/\sqrt{3}} = \sqrt{3}-1$. Squaring this gives us the ratio of areas $\boxed{4-2\sqrt{3}}$.

5. In $\triangle ABC$, $\angle ABC = 75°$ and $\angle BAC$ is obtuse. Points D and E are on AC and BC, respectively, such that $\frac{AB}{BC} = \frac{DE}{EC}$ and $\angle DEC = \angle EDC$. Compute $\angle DEC$ in degrees.

Answer: 85

Solution: Extend AC past A, and draw F on AC such that $AB = FB$. Note that $\frac{AB}{BC} = \frac{FB}{BC} = \frac{DE}{EC}$, and since $\angle FBC = \angle DEC$ we have $\triangle FBC \sim \triangle DEC$.

Next, we perform some angle chasing. Let $\angle DEC = \angle EDC = x$. By similar triangles, we have $\angle FBC = \angle BFC = x$ as well. Furthermore, $FB = AB$, so $\triangle FAB$ is isoceles, and thus $\angle FAB = x$ as well. Now $\angle ABC = 75°$, so we compute $\angle FBA = \angle FBC - \angle ABC = x - 75$. The angles of a triangle sum to $180°$, giving us the equation $3x - 75 = 180$, which solves to $x = \boxed{85}$.

6. In $\triangle ABC$, $AB = 3$, $AC = 6$, and D is drawn on BC such that AD is the angle bisector of $\angle BAC$. D is reflected across AB to a point E, and suppose that AC and BE are parallel. Compute CE.

Answer: $\sqrt{61}$

Solution: Let $\angle BAC = x$. Since $AC||BE$, we have $\angle ABE = x$, so $\angle ABC = x$ since E is the reflection of D across AB. This means that $\triangle ABC$ is isosceles, so $AC = BC = 6$. Using the angle bisector theorem, we find $CD = 4$ and $BD = BE = 2$.

Now let $\theta = \angle CBA$. We can use the Law of Cosines to compute

$$CE = \sqrt{BC^2 + BE^2 - 2\cdot BC\cdot BE\cdot \cos 2\theta}.$$

Because $\triangle ABC$ is isoceles, we can drop the perpendicular from C to find $\cos\theta = \frac{1.5}{6} = \frac{1}{4}$. Using the double angle formula, we get $\cos 2\theta = 2\cos^2\theta - 1 = -\frac{7}{8}$. Plugging this in gives us the answer

$$CE = \sqrt{2^2 + 6^2 - 2\cdot 2\cdot 6\cdot\left(\frac{-7}{8}\right)} = \boxed{\sqrt{61}}.$$

7. Two equilateral triangles ABC and DEF, each with side length 1, are drawn in 2 parallel planes such that when one plane is projected onto the other, the vertices of the triangles form a regular hexagon $AFBDCE$. Line segments AE, AF, BF, BD, CD, and CE are drawn, and suppose that each of these segments also has length 1. Compute the volume of the resulting solid that is formed.

Answer: $\frac{\sqrt{2}}{3}$

Solution 1: Draw lines l_1, l_2, l_3 through A parallel to BC, through B parallel to CA, and through C parallel to AB, respectively. Suppose that l_2 and l_3 intersect at D', l_3 and l_1 intersect at E', and l_1 and l_2 intersect at F'. We observe that DD', EE', and FF' are concurrent at a single point G. In addition, $\triangle D'E'F'$ is an equilateral triangle with side length 2, double that of $\triangle DEF$.

Let the notation $[f]$ denote the volume of a figure f. Then, we observe that $[D'E'F'G] = [ABCDEF] + [ABFF'] + [BCDD'] + [CAEE'] + [DEFG]$. Next, we wish to show that

$DD' = EE' = FF' = 1$; this would allow us to show that $ABFF', BCDD', CAEE', DEFG$ are regular tetrahedra with side length 1, and that $D'E'F'G$ is a regular tetrahedra with side length 2.

Consider the trapezoid $EE'F'F$. Assume for the sake of contradiction that $EE' \neq 1$. Then $FF' \neq 1$ as well, since $EF||E'F'$. Since $AE = AE' = AF = AF' = 1$, both AEE' and AFF' are not equilateral, and $\angle EAE' = \angle FAF'$ is not $60°$. However, this means that $\angle EAF \neq 60°$ since $\angle E'AE + \angle EAF + \angle FAF' = 180°$, contradicting the fact that $\triangle AEF$ is equilateral with side length 1.

Hence, we have $DD' = EE' = FF'$, after extrapolating the previous argument to the third side. Therefore, $[ABCDEF] = [D'E'F'G] - 4[DEFG] = 8[DEFG] - 4[DEFG] = 4[DEFG]$, since a volume which is scaled by twice the side length has its volume scaled by $2^3 = 8$. It suffices to compute the volume of a regular tetrahedron. Dropping the altitude from G to $\triangle DEF$, we can compute the height of the tetrahedron $DEFG$ to be $\frac{\sqrt{6}}{3}$. Hence, the volume of $DEFG$ is

$$\frac{1}{3}\frac{\sqrt{6}}{3}A[\triangle DEF] = \frac{\sqrt{6}}{9}\frac{\sqrt{3}}{4} = \frac{\sqrt{2}}{12}.$$

Finally, our desired area is $4 \times \frac{\sqrt{2}}{12} = \boxed{\frac{\sqrt{2}}{3}}$.

Solution 2: Because each of the edges of the solid have length 1, each of the faces of the solid are equilateral triangles. There are 12 edges, which implies that the solid is in fact a regular octahedron with edge length 1.

To compute the volume of a regular octahedron, we split it into two congruent square pyramids. Let $ABCD$ be the square base with side length 1, and let E be the top of the pyramid. Also, let F be the point in the middle of $ABCD$ and let M be the midpoint of AB. Clearly the base $ABCD$ has area 1. To compute the height EF, we note that $\triangle EFM$ is a right triangle with $FM = \frac{1}{2}$. To compute EM, we note that it is a leg of the right triangle $\triangle AME$ where $AM = \frac{1}{2}$ and $AE = 1$. Using the Pythagorean Theorem on $\triangle AME$ gives us $EM = \frac{\sqrt{3}}{2}$, and using it again on $\triangle EFM$ gives us the height $EF = \frac{\sqrt{2}}{2}$. Therefore, the volume of the square pyramid $ABCDE$ is $\frac{1}{3} \cdot 1 \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{6}$, which implies that the volume of the regular octahedron is $\boxed{\frac{\sqrt{2}}{3}}$.

8. Let ABC be a right triangle with $\angle ACB = 90°$, $BC = 16$, and $AC = 12$. Let the angle bisectors of $\angle BAC$ and $\angle ABC$ intersect BC and AC at D and E respectively. Let AD and BE intersect at I, and let the circle centered at I passing through C intersect AB at P and Q such that $AQ < AP$. Compute the area of quadrilateral $DPQE$.

   Answer: $\mathbf{\frac{136}{3}}$

   Solution: We claim that $AP = AC$. Let P' be a point on AB such that $AP' = AC$. Since $AI = AI$ and $\angle CAI = \angle P'AI$, we have $\triangle ACI \cong \triangle AP'I$ by SAS congruency. Thus, $IC = IP'$, so P' must lie on the circle centered at I passing through C. This implies that either $P = P'$ or $Q = P'$.

   Let the circle centered at I passing through C intersect AC at X. Note that $AX < AC$, and by Power of a Point, $AX \cdot AC = AQ \cdot AP$.

   Now suppose that $P' = Q$. Then $AC = AP' = AQ$, so $AP = AX < AC = AQ$, contradicting $AQ < AP$. Hence, $P' \neq Q$, so $P' = P$ and $AP = AC$.

From here, we notice that $\angle PAD = \angle CAD$ and $AD = AD$, so $\triangle ADP \cong \triangle ADC$ by SAS congruency. As a result, $CD = PD$ and $\angle APD = 90°$. By similar reasoning, we deduce that $CE = QE$ and $\angle BQE = 90°$. It follows that $DPQE$ is a trapezoid. By the Angle Bisector Theorem, we compute that

$$PD = CD = \frac{AC}{AB + AC} \cdot BC = \frac{12}{32} \cdot 16 = 6.$$

Likewise, we compute

$$QE = CE = \frac{BC}{BC + AB} \cdot AC = \frac{16}{36} \cdot 12 = \frac{16}{3}$$

Finally, we have

$$PQ = AP + BQ - AB = AC + BC - AB = 8$$

so the area of $DPQE$ is

$$\frac{1}{2}PQ(QE + PD) = \frac{1}{2} \cdot 8\left(\frac{16}{3} + 6\right) = \boxed{\frac{136}{3}}.$$

9. Let $ABCD$ be a cyclic quadrilateral with $3AB = 2AD$ and $BC = CD$. The diagonals AC and BD intersect at point X. Let E be a point on AD such that $DE = AB$ and Y be the point of intersection of lines AC and BE. If the area of triangle ABY is 5, then what is the area of quadrilateral $DEYX$?

Answer: 11

Solution 1: Let $[A]$ denote the area of polygon A. Since $BC = CD$, $\angle BDC = \angle DBC$. Note that $\angle BAC = \angle BDC$ since they are angles of the same segment and $\angle CAD = \angle CBD$ for the same reason. Hence, $\angle BAC = \angle CAD$ and thus AC is the angle bisector of $\angle BAD$. Therefore,

$$\frac{BX}{XD} = \frac{AB}{AD} = \frac{2}{3}.$$

We also know that

$$\frac{DE}{EA} = \frac{DE}{AD - DE} = \frac{AB}{AD - AB} = 2.$$

Now by Menelaus' Theorem, we have

$$\frac{AY}{YX} \cdot \frac{XB}{BD} \cdot \frac{DE}{EA} = 1.$$

Therefore, $\frac{AY}{YX} = \frac{5}{4}$ so $[BXY] = 4$. Since $\frac{BX}{XD} = \frac{2}{3}$, we have $[DXY] = \frac{3}{2} \cdot 4 = 6$. Now since AY is the angle bisector of $\angle BAE$, we have

$$\frac{BY}{YE} = \frac{AB}{AE} = 2$$

and thus $[AEY] = \frac{5}{2}$. Because $\frac{DE}{EA} = 2$, we have $[DYE] = 2[AEY] = 5$. Finally, we have $[DEYX] = [DEY] + [DYX] = 5 + 6 = \boxed{11}$.

Solution 2: Instead of using Menelaus' Theorem, let the $x = [BXY]$ and $y = [DEYX]$. We know that $[AEY] = \frac{5}{2}$ from above. We then have the two equations

$$\frac{5 + x}{\frac{5}{2} + y} = \frac{2}{3}$$

$$\frac{5 + \frac{5}{2}}{x + y} = \frac{1}{2}.$$

Solving these two equations gives us $x = 4$ and $y = \boxed{11}$.

10. Let ABC be a triangle with $AB = 13$, $AC = 14$, and $BC = 15$, and let Γ be its incircle with incenter I. Let D and E be the points of tangency between Γ and BC and AC respectively, and let ω be the circle inscribed in $CDIE$. If Q is the intersection point between Γ and ω and P is the intersection point between CQ and ω, compute the length of PQ.

Answer: $\frac{8\sqrt{6}}{9}$

Solution: We can derive that $CD = CE = 8$. We then compute the inradius r of $\triangle ABC$. Using Heron's Formula or drawing an altitude from B to AC, we can calculate that the area of $\triangle ABC$ is 84. Since the product of r and the semiperimeter of $\triangle ABC$ also gives the area, we find that $r = 4$.

Let O be the center of ω. Also let ω touch ID at X and CD at Y. Since $OXDY$ is a square, we have $\triangle IXO \simeq \triangle OYC$. Let x be the radius of ω, giving us $XO = YO = x$ and $IX = 4 - x$ and $CY = 8 - x$. This gives us the equation $\frac{4-x}{x} = \frac{x}{8-x}$, and solving for x yields $x = \frac{8}{3}$.

Since both ω and Γ are tangent to AC and BC, a homethety (the enlargement/shrinking of objects with respect to a fixed point and fixed ratio) centered at C sends ω to Γ, and the ratio is $\frac{x}{r} = \frac{2}{3}$. Since C, P, and Q are collinear, the same homothety also takes P to Q, so $\frac{CP}{CQ} = \frac{2}{3}$. Letting $CP = 2k$ and $CQ = 3k$, we have that $PQ = k$. Finally, $CY = 8 - \frac{8}{3} = \frac{16}{3}$, so by Power of a Point,

$$CY^2 = CP \cdot CQ \implies \left(\frac{16}{3}\right)^2 = 6k^2.$$

Solving for k gives us $PQ = \boxed{\frac{8\sqrt{6}}{9}}$.