1. If $f(x) = x^2$ and $g(x) = \ln(x)$, compute $f'(1) + g'(1)$.

   Answer: 3

   Solution: We compute $f'(x) = 2x$ and $g'(x) = \frac{1}{x}$, so plugging in 1 for both gives us $\boxed{3}$.

2. Given that $f(x) = x^2 + ax - 17$, find all real values of a such that $f(4) = f'(4)$.

   Answer: 3

   Solution: Differentiating $f(x)$ gives us $f'(x) = 2x+a$. Now $f(4) = 4a-1$ and $f'(4) = a+8$, so setting the two equal yields $4a - 1 = a + 8$, which solves to $a = \boxed{3}$.

3. Find the value of a such that

$$\int_1^a (3x^2 - 6x + 3)\, dx = 27.$$

   Answer: 4

   Solution: We factor the integrand to get

$$\int_1^a 3(x-1)^2 dx = (x-1)^3\big|_1^a = (a-1)^3 = 27.$$

   Taking the cube root of both sides yields $a - 1 = 3$ or $a = \boxed{4}$.

4. Compute

$$\int_0^4 \frac{dx}{\sqrt{|x-2|}}.$$

   Answer: $4\sqrt{2}$

   Solution: Note that the function $\frac{1}{\sqrt{|x-2|}}$ is discontinuous at $x = 2$. We therefore split the integral into two parts and compute separately:

$$\int_0^2 \frac{dx}{\sqrt{|x-2|}} = \int_0^2 \frac{dx}{\sqrt{2-x}} = -2\sqrt{2-x}\big|_0^2 = 2\sqrt{2}$$
$$\int_2^4 \frac{dx}{\sqrt{|x-2|}} = \int_2^4 \frac{dx}{\sqrt{x-2}} = 2\sqrt{x-2}\big|_2^4 = 2\sqrt{2}$$

   The answer is therefore $2\sqrt{2} + 2\sqrt{2} = \boxed{4\sqrt{2}}$.

5. Eric and Harrison are standing in a field, and Eric is 400 feet directly East of Harrison. Eric starts to walk North at a rate of 4 feet per second, while Harrsion starts to walk South at the same time at a rate of 6 feet per second. After 30 seconds, at what rate is the distance between Eric and Harrison changing?

   Answer: 6

   Solution: We can model the rate of a change as a right triangle with base $x = 400$ feet and height y increasing at a rate of 10 feet per second. After 30 seconds, we will have $y = 300$ feet. If we let z denote the distance between Eric and Harrison, then z is the hypotenuse of the right triangle, giving us the relation $z^2 = x^2 + y^2$. Thus, the distance between Eric and Harrison after 30 seconds is $z = 500$ feet. Differentiating both sides also yields $2z \cdot z' = 2x \cdot x' + 2y \cdot y'$. Plugging in our values, we get $2 \cdot 500z' = 2 \cdot 400 \cdot 0 + 2 \cdot 300 \cdot 10$, which gives us $z' = \boxed{6}$ feet per second.

6. Compute
$$\lim_{x\to 0} \frac{(1-\cos x)^2}{x^2 - x^2\cos^2 x}.$$

Answer: $\mathbf{\frac{1}{4}}$

Solution: Factoring out the x^2 from the denominator and recalling that $\sin^2 x + \cos^2 x = 1$, our limit becomes
$$\lim_{x\to 0} \frac{(1-\cos x)^2}{x^2 \sin^2 x} = \left(\lim_{x\to 0} \frac{1-\cos x}{x\sin x}\right)^2.$$
Note that the limit is in an indeterminate form 0/0 when we plug in $x = 0$, so we apply L'Hopital's rule to get
$$\lim_{x\to 0} \frac{1-\cos x}{x\sin x} = \lim_{x\to 0} \frac{\sin x}{\sin x + x\cos x}.$$
Plugging in $x = 0$ still gives us the indeterminate form 0/0, so we apply L'Hopital's rule again to get
$$\lim_{x\to 0} \frac{\sin x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{\cos x}{\cos x + \cos x - x\sin x}.$$
Plugging in $x = 0$ here gives us $\frac{1}{2}$. Therefore, our answer is $\left(\frac{1}{2}\right)^2 = \boxed{\frac{1}{4}}$.

7. Compute
$$\int_{-2}^{0} \frac{x^3 + 4x^2 + 7x - 20}{x^2 + 4x + 8}\,dx + \int_0^2 \frac{2x^3 - 7x^2 + 9x - 10}{x^2 + 4}\,dx.$$

Answer: 12

Solution: The polynomials in the denominator are similar, so we rewrite them in the form
$$\int_{-2}^{0} \frac{x^3 + 4x^2 + 7x - 20}{(x+2)^2 + 4}\,dx + \int_0^2 \frac{2x^3 - 7x^2 + 9x - 10}{x^2 + 4}\,dx.$$
This suggests we make the u-substution $u = x + 2$ in the first integral, which gives us
$$\int_0^2 \frac{(u-2)^3 + 4(u-2)^2 + 7(u-2) - 20}{u^2 + 4}\,du + \int_0^2 \frac{2x^3 - 7x^2 + 9x - 10}{x^2 + 4}\,dx$$
which simplifies down to
$$\int_0^2 \frac{u^3 - 2u^2 + 3u - 26}{u^2 + 4}\,du + \int_0^2 \frac{2x^3 - 7x^2 + 9x - 10}{x^2 + 4}\,dx.$$
We can then combine the two integrals into a single integral and combine like terms to get
$$\int_0^2 \frac{3x^3 - 9x^2 + 12x - 36}{x^2 + 4}\,dx = \int_0^2 (3x - 9)\,dx = \left(\frac{3}{2}x^2 - 9x\right)\Bigg|_0^2 = \frac{3}{2}(2^2) - 9\cdot 2 = \boxed{-12}.$$

8. Compute
$$\lim_{n\to\infty} n^2 \int_0^{1/n} x^{2018x+1}\,dx.$$

Answer: $\mathbf{\frac{1}{2}}$

Solution: The key to evaluating this limit is to approximate the limit with a simpler integral. In particular, notice that $x^x \to 1$ as $x \to 0$. This suggests that we can approximate the integrand using $x^{2018x} = (x^x)^{2018} \approx 1$ to get

$$\lim_{n\to\infty} n^2 \int_0^{1/n} x^{2018x+1}\,dx = \lim_{n\to\infty} n^2 \int_0^{1/n} x\,dx.$$

The integral then evaluates to $\int_0^{1/n} x\,dx = \frac{1}{2n^2}$. Plugging this back into the limit gives us the answer $\boxed{\frac{1}{2}}$.

To prove this claim rigorously, let $\epsilon > 0$. Because $\lim_{x\to 0} x^{2018x} = 1$, there exists some $\delta > 0$ such that $|x^{2018x} - 1| < \epsilon$ for all $0 < x < \delta$. Now consider the integral

$$n^2 \int_0^{1/n} x^{2018x+1} - x\,dx.$$

Armed with our approximation, so long as $0 < \frac{1}{n} < \delta$, or $n > \frac{1}{\delta}$, we have

$$\begin{aligned}\left|n^2 \int_0^{1/n} x^{2018x+1} - x\,dx\right| &= \left|n^2 \int_0^{1/n} x\left(x^{2018x} - 1\right)\,dx\right| \\ &\le n^2 \int_0^{1/n} x\left|x^{2018x} - 1\right|\,dx \\ &< n^2 \int_0^{1/n} \epsilon \cdot x\,dx \\ &= \frac{\epsilon}{2}\end{aligned}$$

where we have used the facts that the function x is non-negative on the interval $[0, 1/n]$. We have already shown above that $n^2 \int_0^{1/n} x\,dx = \frac{1}{2}$, so for all $n > \frac{1}{\delta}$ we have

$$\left|n^2 \int_0^{1/n} x^{2018x+1}\,dx - \frac{1}{2}\right| < \frac{\epsilon}{2}.$$

This is precisely the definition of a limit, so we conclude that the limit does exist and approaches $\boxed{\frac{1}{2}}$ as claimed above.

9. Compute

$$\int_0^{\pi} \frac{2x\sin x}{3+\cos 2x}\,dx.$$

Answer: $\frac{\pi^2}{4}$

Solution: Let

$$I = \int_0^{\pi} \frac{2x\sin x}{3+\cos 2x}\,dx.$$

We first use the identity $\cos 2x = 2\cos^2 x - 1$ to reduce the integral into the form

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx.$$

Next, we consider the substitution $x \mapsto \pi - y$. The integral then becomes

$$\int_0^{\pi} \frac{(\pi - y)\sin y}{1+\cos^2 y}\,dy.$$

Adding the two integrals together, we get

$$2I = \int_0^\pi \frac{\pi \sin x}{1+\cos^2 x}\, dx.$$

This gives us the simpler

$$I = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\, dx.$$

Now use the substitution $u = \cos x$ to get

$$-\frac{\pi}{2}\int_1^{-1} \frac{1}{1+u^2}\, du.$$

This is the standard $\tan^{-1} u$ integral, so we have

$$I = \frac{\pi}{2}\left(\tan^{-1}(1) - \tan^{-1}(-1)\right) = \frac{\pi}{2}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \boxed{\frac{\pi^2}{4}}.$$

10. *Fact*: The value $\ln(2)$ is not the root of any polynomial with rational coefficients.

For any nonnegative integer n, let $p_n(x)$ be the unique polynomial with integer coefficients such that

$$p_n(\ln(2)) = \int_1^2 (\ln(x))^n\, dx.$$

Compute the value of the sum

$$\sum_{n=0}^{\infty} \frac{1}{p_n(0)}.$$

Answer: $\boldsymbol{\frac{1}{e}}$

Solution: We first attempt to compute the integral $p_n(\ln(2)) = \int_1^2 (\ln(x))^n\, dx$. Using integration by parts with $u = (\ln(x))^n$ and $dv = 1$, we get

$$\begin{aligned}\int_1^2 (\ln(x))^n\, dx &= x(\ln(x))^n\Big|_1^2 - \int_1^2 n(\ln(x))^{n-1}\, dx \\ &= 2(\ln(2))^n - n\int_1^2 (\ln(x))^{n-1}\, dx \\ &= 2(\ln(2))^n - np_{n-1}(\ln(2)).\end{aligned}$$

Because $\ln(2)$ is not the root of any polynomial with rational coefficients, we can plug in $x = \ln(2)$ to get the recurrence

$$p_n(x) = 2x^n - np_{n-1}(x).$$

Recall that we are only concerned with the constant terms $p_n(0)$, so plugging in $x = 0$ yields

$$p_n(0) = -np_{n-1}(0).$$

Now since

$$p_0(\ln(2)) = \int_1^2 dx = 1,$$

we have $p_0(x) = 1$, and therefore $p_0(0) = 1$. From the recurrence, we deduce that

$$p_n(0) = (-1)^n \cdot n!.$$

Our desired sum is thus

$$\sum_{n=0}^{\infty} \frac{1}{p_n(0)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \boxed{\frac{1}{e}}$$

from the Taylor series expansion of e^x at $x = -1$.