1. Compute $\lim_{x\to 3} \dfrac{x^2+2x-15}{x^2-4x+3}$.

   Answer: 4

   Solution: Note that $\dfrac{x^2+2x-15}{x^2-4x+3} = \dfrac{(x-3)(x+5)}{(x-3)(x-1)} = \dfrac{x+5}{x-1}$. Then $\lim_{x\to 3} \dfrac{x+5}{x-1} = \dfrac{3+5}{3-1} = \boxed{4}$.

2. Compute all real values of b such that, for $f(x) = x^2 + bx - 17$, $f(4) = f'(4)$.

   Answer: 3

   Solution: We have that $f(4) = 4b - 1$ and $f'(4) = 2(4) + b = b + 8$. Setting these equal to each other, we see that $b = \boxed{3}$.

3. Suppose a and b are real numbers such that

$$\lim_{x\to 0} \frac{\sin^2 x}{e^{ax} - bx - 1} = \frac{1}{2}.$$

   Determine all possible ordered pairs (a, b).

   Answer: $(2,2)$ and $(-2,-2)$

   Solution: Since this is in an indeterminate form, we can use L'Hôpital's Rule to obtain

$$\lim_{x\to 0} \frac{\sin 2x}{ae^{ax} - b} = \frac{1}{2}.$$

   However, the numerator goes to zero, so the denominator must also go to zero to give us another indeterminate form. This implies that $a = b$. Using L'Hôpital's Rule again, we have that

$$\lim_{x\to 0} \frac{2\cos 2x}{a^2 e^{ax}} = \frac{1}{2}.$$

   The numerator goes to 2, so the denominator must go to 4. Therefore, $a = b = \pm 2$, giving us $(a, b) = \boxed{(2,2) \text{ and } (-2,-2)}$.

4. Evaluate $\displaystyle\int_0^4 e^{\sqrt{x}}\,dx$.

   Answer: $2e^2 + 2$

   Let $w = \sqrt{x}$ so that $w^2 = x$ and $dx = 2w\,dw$. Then the integral becomes $2\displaystyle\int_0^2 we^w\,dw$.

   To find this integral, use integration by parts:

$$\begin{aligned}
u = w \to du = dw; \qquad & dv = e^w\,dw \to v = e^w \\
\int we^w\,dw \quad &= \quad uv - \int v\,du \\
&= \quad we^w - \int e^w dw \\
&= \quad (w-1)e^w.
\end{aligned}$$

   Evaluating $2(w-1)e^w$ at our limits of integration yields $\boxed{2e^2+2}$.

5. Evaluate $\lim\limits_{x\to 0} \dfrac{\sin^2(5x)\tan^3(4x)}{(\log(2x+1))^5}$.

Answer: 50

Solution 1: For any function f with $f(0)=0$, we know that

$$\lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{f(x)-f(0)}{x} = f'(0).$$

$\sin(5x)$, $\tan(4x)$, and $\log(2x+1)$ are all 0 at $x=0$, and their derivatives at 0 are 5, 4, and 2, respectively. So, divide numerator and denominator by x^5 and re-arrange to get

$$\lim_{x\to 0}\frac{\sin^2(5x)\tan^3(4x)}{(\log(2x+1))^5} = \lim_{x\to 0}\frac{\left(\frac{\sin(5x)}{x}\right)^2\cdot\left(\frac{\tan(4x)}{x}\right)^3}{\left(\frac{\log(2x+1)}{x}\right)^5} = \frac{5^2\cdot 4^3}{2^5} = \boxed{50}.$$

Solution 2: Recall from Taylor series that if $f(0)=0$, then $f(x)\approx f'(0)x$ when x is small. This allows us to write

$$\lim_{x\to 0}\frac{\sin^2(5x)\tan^3(4x)}{(\log(2x+1))^5} = \lim_{x\to 0}\frac{(5x)^2(4x)^3}{(2x)^5} = \boxed{50}.$$

6. Compute $\displaystyle\sum_{k=0}^{\infty}\int_0^{\frac{\pi}{3}} \sin^{2k} x\, dx$.

Answer: $\sqrt{3}$

Bring the sum into the integral, so we have

$$\int_0^{\frac{\pi}{3}}\sum_{k=0}^{\infty}\sin^{2k} x\, dx.$$

The integrand is a geometric series, so the answer is

$$\int_0^{\frac{\pi}{3}}\frac{1}{1-\sin^2 x}\,dx = \int_0^{\frac{\pi}{3}}\sec^2 x\,dx = \tan\left(\frac{\pi}{3}\right)-\tan(0) = \boxed{\sqrt{3}}.$$

7. The function $f(x)$ has the property that, for some real positive constant C, the expression

$$\frac{f^{(n)}(x)}{n+x+C}$$

is independent of n for all nonnegative integers n, provided that $n+x+C\neq 0$. Given that $f'(0)=1$ and $\displaystyle\int_0^1 f(x)\,dx = C+(e-2)$, determine the value of C.

Note: $f^{(n)}(x)$ is the n-th derivative of $f(x)$, and $f^{(0)}(x)$ is defined to be $f(x)$.

Answer: $\sqrt{3-e}$

Solution: Since $f^{(n)}(x)/(n+x+C)$ is independent of n, we can say that it is equal to $g(x)$. Multiplying by $(n+x+C)$, we have that

$$f^{(n)}(x) = (n+x+C)g(x).$$

Taking a derivative with respect to x, we obtain

$$f^{(n+1)}(x) = (n + x + C)g'(x) + g(x).$$

However, this is equal to $(n + 1 + x + C)g(x)$ by the problem statement. Canceling terms, we obtain that $g(x) = g'(x)$. The only class of functions that is its own derivative is ae^x, so we have that $g(x) = ae^x$ (for some constant a). Now, $f'(x) = (x + C + 1)ae^x$, so $f'(0) = 1$ gives us that $a = 1/(C + 1)$. We also have that

$$\int_0^1 f(x)\,dx = \int_0^1 \frac{x + C}{C + 1} \cdot e^x\,dx = C + (e - 2).$$

Integration by parts gives us

$$\frac{(e - 1)C + 1}{C + 1} = C + (e - 2),$$

which simplifies to

$$C^2 = 3 - e,$$

from which it follows that the answer is $\boxed{\sqrt{3 - e}}$.

8. The function $f(x)$ is defined for all $x \geq 0$ and is always nonnegative. It has the additional property that if any line is drawn from the origin with any positive slope m, it intersects the graph $y = f(x)$ at precisely one point, which is $\frac{1}{\sqrt{m}}$ units from the origin. Let a be the unique real number for which f takes on its maximum value at $x = a$ (you may assume that such an a exists). Find $\displaystyle\int_0^a f(x)\,dx$.

Answer: $\mathbf{\frac{1+\log(2)}{4}}$

Solution 1: First, express x and y as functions parametrized by m. We have the system

$$y = mx$$
$$x^2 + y^2 = \frac{1}{m}.$$

Solving for y, we get $y = \sqrt{\frac{m}{1+m^2}}$. Hence, maximizing y is equivalent to maximizing $\frac{m}{1+m^2}$. By differentiating with respect to m, we see that the maximum occurs when $m = 1$, at the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Now, we just need to compute the integral. However, this parametric form is not convenient. Instead, by drawing the line $y = x$, we notice that the integral splits up into a right isosceles triangle, and a region between the line $y = x$ and the y-axis. This suggests that we should convert to polar coordinates. In fact, $f(x)$ is equivalent to the graph $r(\theta) = \frac{1}{\sqrt{\tan(\theta)}}$, since a line at angle θ to the x-axis has slope $\tan(\theta)$. The area we wish to compute is

$$\begin{aligned}\int_{\pi/4}^{\pi/2} \frac{1}{2} r(\theta)^2\,d\theta &= \frac{1}{2}\int_{\pi/4}^{\pi/2} \cot(\theta)\,d\theta \\ &= \frac{1}{2}\left[\log(\sin(\theta))\right]_{\pi/4}^{\pi/2} \\ &= \frac{1}{2}(0 - \log(1/\sqrt{2})) \\ &= \frac{1}{4}\log(2).\end{aligned}$$

We add this area to the area of the triangle, which is $\frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{4}$, so our final answer is $\boxed{\dfrac{1+\log(2)}{4}}$.

Solution 2: We begin as before to find a, but present a different method of computing the integral.

Solving for x in terms of y, we get that

$$x^2 + y^2 = x/y \implies yx^2 - x + y^3 = 0 \implies x = \frac{1 \pm \sqrt{1-4y^4}}{2y}.$$

We only care about the region where $1 \pm \sqrt{1-4y^4} = 2xy \le 1$, since $x, y \le \frac{1}{\sqrt{2}}$. Hence, we take $x = \frac{1-\sqrt{1-4y^4}}{2y}$.

Notice that we can compute the desired quantity as

$$\left(\frac{1}{\sqrt{2}}\right)^2 - \int_0^{\frac{1}{\sqrt{2}}} \frac{1-\sqrt{1-4y^4}}{2y}\,dy,$$

since within the square bounded by the coordinate axes and $x, y \le \frac{1}{\sqrt{2}}$, the area between the curve and the x-axis plus the area between the curve and the y-axis sum to the area of the whole square.

Now, using the substitution $u = \sqrt{1-4y^4}$, we get

$$\begin{aligned}
\int_0^{\frac{1}{\sqrt{2}}} \frac{1-\sqrt{1-4y^4}}{2y}\,dy &= \int_1^0 \frac{(1-u)u}{4u^2-4}\,du \\
&= \frac{1}{4}\int_0^1 \frac{(1-u)u}{(1-u)(1+u)}\,du \\
&= \frac{1}{4}\int_0^1 \frac{u}{1+u}\,du \\
&= \frac{1}{4}\int_0^1 1 - \frac{1}{1+u}\,du \\
&= \frac{1}{4}\left[u - \log(1+u)\right]_0^1 \\
&= \frac{1-\log(2)}{4}.
\end{aligned}$$

The answer is $\frac{1}{2}$ minus this quantity, so report $\boxed{\dfrac{1+\log(2)}{4}}$.

9. Evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{\left(\sqrt{\sin x} + \sqrt{\cos x}\right)^4}$.

Answer: 1/3

Solution 1: Observe that by pulling a factor of $\cos^2 x$ out of the denominator, we can write the given integral as

$$\int_0^{\pi/2} \frac{dx}{(1+\sqrt{\tan x})^4 \cos^2 x} = \int_0^{\pi/2} \frac{\sec^2 x\,dx}{(1+\sqrt{\tan x})^4}.$$

We now substitute $u = \sqrt{\tan x} + 1$:

$$du = \frac{\sec^2 x}{2\sqrt{\tan x}}\,dx = \frac{\sec^2 x}{2(u-1)}\,dx.$$

Thus, our integral is equal to

$$\int_1^\infty \frac{2u-2}{u^4}\,du = \int_1^\infty 2u^{-3} - 2u^{-4}\,du,$$

which simplifies to

$$\left[-u^{-2} + \frac{2}{3}u^{-3}\right]_1^\infty = \boxed{\frac{1}{3}}.$$

Solution 2: Let I be the value of the given integral. Note that

$$\frac{1}{2}I = \frac{1}{2}\int_0^{\pi/2}\left(\left(\sqrt{\sin(x)} + \sqrt{\cos(x)}\right)^{-2}\right)^2 dx,$$

which is the polar area bounded by the curve $r(\theta) = \left(\sqrt{\sin(\theta)} + \sqrt{\cos(\theta)}\right)^{-2}$ and the x and y axes for $\theta \in [0, \pi/2]$. Converting to Cartesian coordinates, we get

$$\begin{aligned} 1 &= r\left(\sqrt{\sin(\theta)} + \sqrt{\cos(\theta)}\right)^2 \\ &= \left(\sqrt{r\sin(\theta)} + \sqrt{r\cos(\theta)}\right)^2 \\ &\implies \sqrt{x} + \sqrt{y} = 1 \\ &\implies y = (1-\sqrt{x})^2 = 1 + x - 2\sqrt{x}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{2}I &= \int_0^1 1 + x - 2\sqrt{x}\,dx \\ &= \left[x + \frac{x^2}{2} - \frac{4}{3}x^{3/2}\right]_0^1 \\ &= 1 + \frac{1}{2} - \frac{4}{3} = \frac{1}{6} \\ &\implies I = \boxed{\frac{1}{3}}. \end{aligned}$$

10. Evaluate $\lim_{n\to\infty}\left[\left(\prod_{k=1}^n \frac{2k}{2k-1}\right)\int_{-1}^\infty \frac{(\cos x)^{2n}}{2^x}\,dx\right]$.

Answer: $\boldsymbol{\pi\frac{2^\pi}{2^\pi - 1}}$

Solution 1: Observe that $(\cos x)^{2n}$ looks like a bunch of spikes, centered at $0, \pi, 2\pi, \ldots$, each with area $I_n = \int_{-\pi/2}^{\pi/2}(\cos x)^{2n}\,dx$.

We can integrate by parts to see that

$$I_k = \int_{-\pi/2}^{\pi/2} (\cos x)^{2k}\,dx = \Big[(\cos x)^{2k-1}\sin x\Big]_{-\pi/2}^{\pi/2} + (2k-1)\int_{-\pi/2}^{\pi/2} (\cos x)^{2k-2}\sin^2 x\,dx$$
$$= (2k-1)\int_{-\pi/2}^{\pi/2} (\cos x)^{2k-2}(1-\cos^2 x)\,dx = (2k-1)(I_{k-1}-I_k).$$

Therefore,

$$I_k = \frac{2k-1}{2k} I_{k-1} \implies I_n = \left(\prod_{k=1}^{n} \frac{2k-1}{2k}\right) I_0 = \pi \prod_{k=1}^{n} \frac{2k-1}{2k}.$$

As $n \to \infty$, the spikes get sharper and sharper; this means that the denominator 2^x of the integrand gets concentrated at $x = 0, \pi, 2\pi, \ldots$. Therefore, we expect that as $n \to \infty$,

$$\left(\prod_{k=1}^{n} \frac{2k}{2k-1}\right) \int_{-1}^{\infty} \frac{(\cos x)^{2n}}{2^x}\,dx \to \left(\prod_{k=1}^{n} \frac{2k}{2k-1}\right) \sum_{k=0}^{\infty} \frac{I_n}{2^{k\pi}} = \pi \frac{1}{1-2^{-\pi}} = \boxed{\pi \frac{2^\pi}{2^\pi - 1}}.$$

Solution 2: We present a more rigorous approach here. First, rewrite the problem into the following form:

For each positive integer n, let $a_n = \int_{-1}^{\infty} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\,dx$. Additionally, let $c = \lim_{n\to\infty} \sqrt{n} \prod_{k=1}^{n} (1 - \frac{1}{2k})$, which is a positive finite constant. Evaluate $\frac{1}{c} \lim_{n\to\infty} a_n$.

Let $B = \{0, \pi, 2\pi, \ldots\}$. The idea is that the numerator of the integrand approaches a function with $c\pi$ area concentrated infinitely closely to each point in B. Therefore, the limit should be

$$\lim_{n\to\infty} \int_{-1}^{\infty} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\,dx = \sum_{x\in B} \frac{c\pi}{2^x} = c\pi \frac{2^\pi}{2^\pi - 1},$$

where the last equality follows by the formula for summing geometric series.

We will soon get to a more precise way of thinking about the area being concentrated infinitely closely to points in B, but first let's see why the numerator should have $c\pi$ area around each point in B. Since the area around each point in B is the same ($\cos^{2n}$ is periodic), we need only consider the area around 0. We can apply integration by parts to find a formula for the area around 0 in each term of the sequence. The recurrence is

$$\int_{-\pi/2}^{\pi/2} \sqrt{n}(\cos x)^{2n}\,dx = (1 - \frac{1}{2n}) \int_{-\pi/2}^{\pi/2} \sqrt{n}(\cos x)^{2(n-1)}\,dx.$$

Repeatedly applying this formula, we get

$$\int_{-\pi/2}^{\pi/2} \sqrt{n}(\cos x)^{2n}\,dx = \pi\sqrt{n} \prod_{k=1}^{n} \left(1 - \frac{1}{2k}\right).$$

Taking the limit as $n \to \infty$, the area around 0 goes to $c\pi$. So the answer makes sense. Now we will prove it more rigorously.

For $x \notin B$, the integrand $\frac{\sqrt{n}(\cos x)^{2n}}{2^x}$ goes to 0 as $n \to \infty$ because $\cos x < 1$. So for any open set S sufficiently disjoint from B, we might guess that

$$\lim_{n\to\infty} \int_S \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\,dx = 0.$$

If we require that every point in S is at least $\epsilon > 0$ away from any point in B, then this is indeed true. There are two ways to see this.

The fanciest way to see it is to use the "dominated convergence theorem", which says that if a sequence of functions f_n converges pointwise to a funciton f and if there is some function φ with $|f_n(x)| < \varphi(x)$ for all $x \in S$ and $\int_S \varphi < \infty$, then $\lim_{n\to\infty} \int_S f_n = \int_S f$. To apply this theorem, we let f_n be the integrand of the n-th term of the sequence. To construct φ, notice that since every point in S is at least ϵ away from every point in B, there is some $\delta < 1$ so that $|\cos x| < \delta$ for all $x \in S$. So $\sqrt{n}(\cos x)^{2n}$ is bounded by $\sqrt{n}\delta^{2n}$ for all $x \in S$. Since $\sqrt{n}\delta^{2n}$ has a finite limit as $n \to \infty$, $\sqrt{n}(\cos x)^{2n}$ is bounded by some finite number B for all $x \in S$. So we can let $\varphi(x) = B/2^x$. Then $|f_n(x)| < \varphi(x)$ for all $x \in S$ and $\int_S \varphi < \infty$, just as we need in order to apply the theorem. So we apply the theorem to get

$$\lim_{n\to\infty} \int_S \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\,dx = \int_S \lim_{n\to\infty} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\,dx = \int_S 0\,dx = 0.$$

But we of course don't expect you to know the dominated convergence theorem, so we can also prove this using a "bare hands" method that is actually easier. (Bare hands is usually much harder than the dominated convergence theorem proof. That is why people use the dominated convergence theorem. But we have arranged for this problem to work with bare hands.) As we argued above, there is some $\delta < 1$ so that $\cos x < \delta$ for all $x \in S$. Then $\sqrt{n}(\cos x)^{2n} < \sqrt{n}\delta^{2n}$ for all $x \in S$. So we have a bound

$$\int_S \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\,dx \le \sqrt{n}\delta^{2n} \int_S \frac{1}{2^x}\,dx.$$

Since the integral $\int \frac{1}{2^x}$ converges, this goes to 0 as $n \to \infty$ so we again have the desired result.

The upshot of all this is that we can now define B_ϵ to be the points in $[-1, \infty)$ that are within ϵ of B and have

$$\lim_{n\to\infty} \int_{-1}^{\infty} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\,dx = \lim_{n\to\infty} \int_{B_\epsilon} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\,dx.$$

To calculate the integral on the right, notice that it is just the sum over all integers $k \ge 0$ of

$$\int_{k\pi-\epsilon}^{k\pi+\epsilon} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\,dx = \frac{1}{2^{k\pi}} \int_{-\epsilon}^{\epsilon} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\,dx.$$

By the formula for summing geometric series, the sum of this over all integers $k \ge 0$ is

$$\lim_{n\to\infty} \int_{-1}^{\infty} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\,dx = \frac{2^\pi}{2^\pi - 1} \int_{-\epsilon}^{\epsilon} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\,dx. \tag{1}$$

So we have reduced the problem to calculating the following limit:

$$\lim_{n\to\infty} \int_{-\epsilon}^{\epsilon} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\,dx.$$

To do this, bound the limit above and below by taking the highest and lowest possible values of 2^x out of the integral:

$$2^{-\epsilon}\lim_{n\to\infty}\int_{-\epsilon}^{\epsilon}\sqrt{n}(\cos x)^{2n}\,dx \le \lim_{n\to\infty}\int_{-\epsilon}^{\epsilon}\frac{\sqrt{n}(\cos x)^{2n}}{2^x}\,dx \le 2^{\epsilon}\lim_{n\to\infty}\int_{-\epsilon}^{\epsilon}\sqrt{n}(\cos x)^{2n}\,dx.$$

By a very similar argument as above, nothing outside $(-\epsilon,\epsilon)$ contributes to the integrals in our bounds and therefore

$$\lim_{n\to\infty}\int_{-\epsilon}^{\epsilon}\sqrt{n}(\cos x)^{2n}\,dx = \lim_{n\to\infty}\int_{-\pi/2}^{\pi/2}\sqrt{n}(\cos x)^{2n}\,dx.$$

We have already calculated the right hand side: it is $c\pi$. So we can plug this back into our bounds to get

$$2^{-\epsilon}c\pi \le \lim_{n\to\infty}\int_{-\epsilon}^{\epsilon}\frac{\sqrt{n}(\cos x)^{2n}}{2^x}\,dx \le 2^{\epsilon}c\pi.$$

Plugging this bound into (1) gives

$$2^{-\epsilon}c\pi\frac{2^\pi}{2^\pi-1} \le \lim_{n\to\infty}\int_{-1}^{\infty}\frac{\sqrt{n}(\cos x)^{2n}}{2^x}\,dx \le 2^{\epsilon}c\pi\frac{2^\pi}{2^\pi-1}.$$

Since ϵ was arbitrary, taking $\epsilon\to 0$ forces

$$\lim_{n\to\infty}\int_{-1}^{\infty}\frac{\sqrt{n}(\cos x)^{2n}}{2^x}\,dx = c\pi\frac{2^\pi}{2^\pi-1},$$

as desired.

Finally, you might be interested in knowing why c is a positive finite constant. (This is not necessary to solve the problem, but it is necessary to be sure that the problem makes sense. And it is interesting.) To see this, let $b_n = \sqrt{n}\prod_{k=1}^{n}(1-\frac{1}{2k})$. Then

$$\log b_n = \frac{1}{2}\log n + \sum_{k=1}^{n}\log\left(1-\frac{1}{2k}\right) = \frac{1}{2}\log n + \sum_{k=1}^{n}\left(-\frac{1}{2k}+O\left(\frac{1}{k^2}\right)\right).$$

Here $O(\frac{1}{k^2})$ denotes some function of k whose absolute value is always less than $C\frac{1}{k^2}$ for some C big enough. The fact that $\log(1-\frac{1}{2k}) = -\frac{1}{2k}+O(\frac{1}{k^2})$ follows from the Taylor expansion of log around 1. It is well known that $\sum_{k=1}^{n}\frac{1}{k} = \log n + \gamma + O(\frac{1}{n})$ where γ is some constant. Plugging this in gives

$$\log b_n = \frac{1}{2}\log n - \frac{1}{2}\log n - \frac{1}{2}\gamma + O\left(\frac{1}{n}\right) + \sum_{k=1}^{n}O\left(\frac{1}{k^2}\right) = -\frac{1}{2}\gamma + O\left(\frac{1}{n}\right) + \sum_{k=1}^{n}O\left(\frac{1}{k^2}\right).$$

Since $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges to some constant, $\sum_{k=1}^{n}O(\frac{1}{k^2})$ converges to some finite constant α as $n\to\infty$. Therefore $\log b_n \to -\frac{1}{2}\gamma+\alpha$ as $n\to\infty$. This is some finite number, so $c = \exp(-\frac{1}{2}\gamma+\alpha)$ is a positive finite number, as desired.