1. How many functions $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ take on exactly 3 distinct values?

   Answer: 1500

   Solution: There are $\binom{5}{3}$ possibilities for the range, so the answer is $10N$ where N is the number of surjective functions from $\{1, 2, 3, 4, 5\}$ to a given 3-element set. The total number of functions $\{1, 2, ..., 5\} \to \{1, 2, 3\}$ is 3^5, from which we subtract $\binom{3}{2}$ (the number of 2-element subsets of $\{1, 2, 3\}$) times 2^5 (the number of functions mapping into that subset), but then (according to the Principle of Inclusion-Exclusion) we must add back $\binom{3}{1}$ (the number of functions mapping into a 1-element subset of $\{1, 2, 3\}$). Thus: $N = 3^5 - \binom{3}{2}(2^5) + \binom{3}{1}(1^5) = 150$. So $10N = 10(150) = \boxed{1500}$.

2. Let i be one of the numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$. Suppose that for all positive integers n, the number n^n never has remainder i upon division by 12. List all possible values of i.

   Answer: 2, 6, 8, 10

   Solution: The table below gives the value of $k^n \pmod{12}$ for $k = 0, 1, \ldots, 11$ and $n = 1, 2, 3$. (Note that when $n = 1$, this is just the value of k.)

k	$n = 2$	$n = 3$
0	0	0
1	1	1
2	4	8
3	9	3
4	4	4
5	1	5
6	0	0
7	1	7
8	4	8
9	9	9
10	4	4
11	1	11

   All values of k have period 1 or 2 (the rows for 2 and 8 continue 4, 8, 4, 8, etc.) We can see that n^n cannot be congruent to $2, 6, 10$ when divided by 12 for $n > 1$, and hence cannot be congruent to $2, 6, 10$ at all. For n^n to be congruent to $8 \pmod{12}$, we would need either $n \equiv 2 \pmod{12}$ and $n \equiv 1 \pmod 2$ or $n \equiv 8 \pmod{12}$ and $n \equiv 1 \pmod 2$; this is impossible since a number which is 2 or 8 (mod 12) must be even. All other remainders indeed occur; this can be checked by inspection, with the help of the Chinese Remainder Theorem. So our answer is $\boxed{2, 6, 8, 10}$.

3. A *card* is an ordered 4-tuple (a_1, a_2, a_3, a_4) where each a_i is chosen from $\{0, 1, 2\}$. A *line* is an (unordered) set of three (distinct) cards $\{(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4)\}$ such that for each i, the numbers a_i, b_i, c_i are either all the same or all different. How many different lines are there?

   Answer: 1080

   Solution: There are $3^4 = 81$ different cards. Any choice of two cards determines a unique line, since if we know a_i and b_i, c_i must equal a_i, b_i if $a_i = b_i$ and must equal neither if $a_i \neq b_i$. This produces $\binom{81}{2}$ lines, but for each line there are 3 pairs of cards which generate that line, so our final answer is $\frac{1}{3}\binom{81}{2} = 27 \cdot 40 = \boxed{1080}$.

4. We say that the pair of positive integers (x, y), where $x < y$, is a *k-tangent pair* if we have $\arctan\dfrac{1}{k} = \arctan\dfrac{1}{x} + \arctan\dfrac{1}{y}$. Compute the second largest integer that appears in a 2012-tangent pair.

   Answer: 811641

   Solution: By taking tangents of both sides we have

$$\frac{1}{k} = \frac{1/x + 1/y}{1 - 1/xy} = \frac{y + x}{xy - 1},$$

   so $xy = k(x + y) + 1$, $(x - k)(y - k) = k^2 + 1$. For $k = 2012$ the second largest factor of $k^2 + 1$ is $(k^2 + 1)/5 = 809629$ and thus the second largest integer y is $k + 809629 = \boxed{811641}$.

5. Regular hexagon $A_1A_2A_3A_4A_5A_6$ has side length 1. For $i = 1, \ldots, 6$, choose B_i to be a point on the segment A_iA_{i+1} uniformly at random, assuming the convention that $A_{j+6} = A_j$ for all integers j. What is the expected value of the area of hexagon $B_1B_2B_3B_4B_5B_6$?

   Answer: $\frac{9\sqrt{3}}{8}$

   Solution 1: By symmetry, $E[(B_iA_{i+1}B_{i+1})] = E[(B_jA_{j+1}B_{j+1})]$ for all integers i and j. Therefore, applying linearity of expectation, the expected area of $B_1B_2B_3B_4B_5B_6$ is equal to the area of $A_1A_2A_3A_4A_5A_6$ minus six times the expected area of $B_1A_2B_2$. Since the lengths of $\overline{B_1A_2}$ and $\overline{B_2A_2}$ are independent, this expectation is equal to $(\frac{1}{2}\sin 120°)E[B_1A_2]E[B_2A_2]$. It is easy to see that $E[B_1A_2] = E[B_2A_2] = 1/2$, so $E[(A_1B_2A_2)] = \frac{\sqrt{3}}{16}$. The area of a unit regular hexagon is $6(\sqrt{3}/4)$, so our answer is $6(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{16}) = \boxed{\dfrac{9\sqrt{3}}{8}}$.

   Solution 2: Since the area of $B_1B_2B_3B_4B_5B_6$ is linear in the location of B_i for each i, and the B_i are all independent, we can argue that the average case comes when each B_i is a midpoint of A_iA_{i+1}.

6. Evaluate

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm(n + m + 1)}.$$

   Answer: 2

   Solution 1: Using the partial fraction

$$\frac{1}{m(n + m + 1)} = \frac{1}{n + 1}\left(\frac{1}{m} - \frac{1}{n + m + 1}\right)$$

   we can sum the series in m first to get

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm(n + m + 1)} &= \sum_{n=1}^{\infty}\frac{1}{n(n + 1)}\left(\frac{1}{1} + \frac{1}{2} + \cdots \frac{1}{n + 1}\right) \\ &= \sum_{k \le n+1}\frac{1}{kn(n + 1)}.\end{aligned}$$

For $k = 1$, this sum has a term for each $n \geq 1$, and for larger values of k, it has a term for each $n \geq k - 1$. Then we can rewrite this as

$$= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{k=2}^{\infty} \frac{1}{k} \sum_{n=k-1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= 1 + \sum_{k=2}^{\infty} \frac{1}{k(k-1)} = 1 + 1 = \boxed{2}.$$

Solution 2: Consider the Taylor expansion

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

We square this equation to obtain

$$(\log(1-x))^2 = \sum_{n,m=1}^{\infty} \frac{x^{n+m}}{nm}.$$

Integrating both sides from 0 to 1 will give the answer. We make the substitution

$$x = 1 - e^y, 0 \geq y > -\infty$$

to get

$$\int_0^1 (\log(1-x))^2 dx = \int_0^{-\infty} (\log e^y)^2 (-e^y dy) = \int_0^{\infty} y^2 e^{-y} dy = \boxed{2}$$

upon integrating by parts twice.

7. A plane in 3-dimensional space passes through the point (a_1, a_2, a_3), with a_1, a_2, and a_3 all positive. The plane also intersects all three coordinate axes with intercepts greater than zero (i.e. there exist positive numbers b_1, b_2, b_3 such that $(b_1, 0, 0)$, $(0, b_2, 0)$, and $(0, 0, b_3)$ all lie on this plane). Find, in terms of a_1, a_2, a_3, the minimum possible volume of the tetrahedron formed by the origin and these three intercepts.

Answer: $\boldsymbol{\frac{9}{2}a_1a_2a_3}$

Solution: Let the x, y, and z intercepts of the plane be b_1, b_2, and b_3, respectively. The tetrahedron in question has volume $\frac{1}{6}b_1b_2b_3$. The equation of our plane is $\frac{x}{b_1} + \frac{y}{b_2} + \frac{z}{b_3} = 1$, since these three intercepts determine the plane. Therefore, we are minimizing $\frac{1}{6}b_1b_2b_3$ subject to the constraint $\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} = 1$. By AM-GM, we get that

$$\frac{a_1a_2a_3}{b_1b_2b_3} \leq \left(\frac{\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3}}{3}\right)^3 = \frac{1}{27}.$$

Therefore, $b_1b_2b_3 \geq 27a_1a_2a_3$ with equality if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \frac{1}{3},$$

which is attained by choosing $b_i = 3a_i$. Hence, the desired minimum volume is $\frac{27a_1a_2a_3}{6} = \boxed{\frac{9}{2}a_1a_2a_3}$.

8. The left end of a rubber band e meters long is attached to a wall and a slightly sadistic child holds on to the right end. A point-sized ant is located at the left end of the rubber band at time $t = 0$, when it begins walking to the right along the rubber band as the child begins stretching it. The increasingly tired ant walks at a rate of $1/(\ln(t + e))$ *centimeters* per second, while the child uniformly stretches the rubber band at a rate of one *meter* per second. The rubber band is infinitely stretchable and the ant and child are immortal. Compute the time in seconds, if it exists, at which the ant reaches the right end of the rubber band. If the ant never reaches the right end, answer $+\infty$.

   Answer: $\boldsymbol{e^{e^{100}} - e}$

   Solution: Let $x(t)$ be the position of the ant at time t, $l(t)$ be the length of the rubber band at time t, and $v_a(t) = \frac{1}{100\log(t+e)}$ be the ant's walking speed relative to the rubber band. We have $\frac{dx}{dt} = v_a(t) + x \cdot \frac{dl/dt}{l(t)}$, where the second term comes from the stretching of the rubber band (the rate of change in length divided by the length gives the rate of stretching of an infinitesimal piece of rubber band; multiplying by $x(t)$ gives the contribution to the ant's forward motion from the part of the rubber band behind the ant). This is just $\frac{dx}{dt} = v_a(t) + x\frac{d}{dt}(\log l(t))$, and multiplying both sides by $e^{-\log l(t)}$ and rearranging, we have $\frac{d}{dt}\left(\frac{x}{l(t)}\right) = \frac{v_a(t)}{l(t)}$, or $\frac{x(t)}{l(t)} = \int_0^t \frac{v_a(s)}{l(s)}\,ds$ (applying initial conditions). We need $\int_0^t \frac{v_a(s)}{l(s)}\,ds = \frac{1}{100}\int_0^t \frac{1}{(s+e)\log(s+e)}\,ds = \frac{1}{100}[\log\log(s+e)]_0^t$ to equal 1, which occurs at time $t = \boxed{e^{e^{100}} - e}$.

9. We say that two lattice points are *neighboring* if the distance between them is 1. We say that a point lies at distance d from a line segment if d is the minimum distance between the point and any point on the line segment. Finally, we say that a lattice point A is *nearby* a line segment if the distance between A and the line segment is no greater than the distance between the line segment and any neighbor of A. Find the number of lattice points that are nearby the line segment connecting the origin and the point $(1984, 2012)$.

   Answer: 1989

   Solution: For notational convenience, let ℓ be the line passing through the origin and $(1984, 2012)$.

   First, note that a point P can only be nearby the given line segment if its x-coordinate is between 0 and 1984, inclusive. If the x-coordinate of P is negative, its distance to any point on the line segment is less than the distance between that point on the line segment and the point one unit to the right of P; if P has x-coordinate greater than 1984, take the point one unit to the left. This inequality holds due to the creation of a right or obtuse angle between P, the point next to P, and any point on the line segment (the edge case where the three points are collinear remains, but this is easily checked separately).

   Now, fix a value $x \in \{0, 1, \ldots, 1984\}$, and let S be the set of lattice points with that x coordinate. Let Q be the point on the line segment with this x-coordinate. Note that for any $P \in S$, the distance between P and the line segment is either the distance from P to ℓ or the distance from P to one of the endpoints of the line segment. In the latter case, since there always exists a cardinal direction to move closer to a given point, P is not nearby our segment. Now consider the former case. By similar triangles, the distance between P and ℓ is proportional to PQ, and so the only P which could possibly be nearby the segment are the P closest to Q. There are two such points if the y-coordinate of Q has fractional part $1/2$, and one such point otherwise.

   Finally, we show that all such points are in fact nearby: this relies on the fact that the slope of ℓ is greater than 1. Consider a point P that is no further from ℓ than any other lattice point with the same x-coordinate. We already know that its distance to Q, the point on the line with same x-coordinate, is less than or equal to $1/2$. Now draw the horizontal line through P, intersecting ℓ at R. Since the slope of ℓ is greater than 1, we have $PR < PQ \leq 1/2$, and so P is, out of all lattice points with the same y coordinate, the closest one to ℓ. Hence, it is a nearby point.

Now, we just need to count the number of nearby points. There are 1985 different valid choices of x-coordinate, and we must double-count all the ones for which the point on ℓ with that x-coordinate has y-coordinate with real part $1/2$. Since ℓ is given by

$$y = \frac{2012}{1984}x = \frac{503}{496}x,$$

this condition holds when $x \equiv 248 \pmod{496}$, so there are 4 such values in the relevant interval. Hence, report $\boxed{1989}$.

10. A permutation of the first n positive integers is *valid* if, for all $i > 1$, i comes after $\left\lfloor \frac{i}{2} \right\rfloor$ in the permutation. What is the probability that a random permutation of the first 14 integers is valid?

Answer: $\frac{1}{31752}$

Solution 1: 1 must be the first number in the permutation; this happens with probability $\frac{1}{14}$.

2 must come before 4, 5, 8, 9 10, and 11; this happens with probability $\frac{1}{7}$.

3 must come before 6, 7, 12, 13, and 14; this happens with probability $\frac{1}{6}$.

4 must come before 8 and 9; this happens with probability $\frac{1}{3}$.

5 must come before 10 and 11; this happens with probability $\frac{1}{3}$.

6 must come before 12 and 13; this happens with probability $\frac{1}{3}$.

7 must come before 14; this happens with probability $\frac{1}{2}$.

All these events are independent, so the answer is the product of the above probabilities, or $\boxed{\frac{1}{31752}}$.

Solution 2: Create a directed graph with vertices labeled 1 through 14, with an arrow from vertex a to vertex b if b has to come after a in a valid permutation. The graph looks like this: 1 points to 2 and 3, 2 points to 4 and 5, 3 points to 6 and 7, 4 points to 8 and 9, 5 points to 10 and 11, 6 points to 12 and 13, 7 points to 14. We want to count the number of ways to merge this graph into a line. There are two ways of ordering 12 and 13 WRT 6. Once this is fixed, there are $\binom{5}{3} = 10$ ways of ordering the $6-12-13$ group and the $7-14$ group WRT 3. Similarly, there are 2 ways of ordering $8, 9$ WRT 4, 2 ways of ordering $10, 11$ WRT 5, and $\binom{6}{3} = 20$ ways of ordering the $4-8-9$ group and the $5-10-11$ group WRT 2. Finally, there are $\binom{13}{6}$ ways of ordering the $2\ldots$ group and the $3\ldots$ group WRT 1. This gives $2 \cdot 10 \cdot 2 \cdot 2 \cdot 20 \cdot \binom{13}{6}$ valid permutations, and dividing by 14! gives the answer.

11. Given that $x, y, z > 0$ and $xyz = 1$, find the range of all possible values of

$$\frac{x^3 + y^3 + z^3 - x^{-3} - y^{-3} - z^{-3}}{x + y + z - x^{-1} - y^{-1} - z^{-1}}.$$

Answer: $(27, +\infty)$

Solution: We make the following modifications to the numerator. Since $xyz = 1$, we may multiply x^{-3}, y^{-3}, z^{-3} by $x^3y^3z^3$, and also add $-1 + x^3y^3z^3$. The numerator then factors as $-1 + x^3 + y^3 + z^3 - x^3y^3 - x^3z^3 - y^3z^3 + x^3y^3z^3 = (x^3 - 1)(y^3 - 1)(z^3 - 1)$. Similarly, the denominator factors as $(x-1)(y-1)(z-1)$, so that the expression can be rewritten as $(x^2+x+1)(y^2 + y + 1)(z^2 + z + 1)$. Dividing this by xyz and writing $z = 1/(xy)$, we find that this is in fact $\left(1 + x + \frac{1}{x}\right)\left(1 + y + \frac{1}{y}\right)\left(1 + xy + \frac{1}{xy}\right)$. We have from $(a-1)^2 \geq 0$ that $a + 1/a \geq 2$, so the product has value at least $(1+2)(1+2)(1+2) = 27$. Equality cannot occur: this would require $x = 1/x, y = 1/y$, or $x = y = z = 1$, making the original denominator zero. Everything larger than 27 can occur, however: we simply consider the special case $y = x$, when the expression reduces to $\left(1 + x + \frac{1}{x}\right)\left(1 + x + \frac{1}{x}\right)\left(1 + x^2 + \frac{1}{x^2}\right)$, a continuous function of x which is unbounded as $x \to +\infty$. Thus we answer $\boxed{(27, +\infty)}$.

12. A triangle has sides of length $\sqrt{2}$, $3 + \sqrt{3}$, and $2\sqrt{2} + \sqrt{6}$. Compute the area of the smallest regular polygon that has three vertices coinciding with the vertices of the given triangle.

Answer: $12 + 6\sqrt{3}$

Solution: We begin by computing the angles of the triangle; repeated application of the Law of Cosines gives us that the angles of the triangle are $15°$, $60°$, and $105°$. Therefore, since the greatest common divisor of the angles of the triangle is $15°$, a dodecagon is the smallest regular polygon that satisfies our current constraints. In particular, the dodecagon has side length $\sqrt{2}$, and it remains to compute the area of the dodecagon. The area of a regular dodecagon is $3\cot\left(\frac{\pi}{12}\right)(\sqrt{2})^2 = \boxed{12 + 6\sqrt{3}}$.

13. How many positive integers n are there such that for any natural numbers a, b, we have $n \mid (a^2b + 1)$ implies $n \mid (a^2 + b)$? (Note: The symbol $\mid$ means "divides"; if $x \mid y$ then y is a multiple of x.)

Answer: 20

Solution: Let $\mathcal{P}$ represent the property of n such that $n \mid a^2b + 1 \Rightarrow n \mid a^2 + b$ for all $a, b \in \mathbb{N}$. Let $\mathcal{Q}$ represent the property of n such that $(a, n) = 1 \Rightarrow n \mid a^4 - 1$ for all $a \in \mathbb{N}$. We shall prove that they are equivalent.

Proof that $\mathcal{P} \Rightarrow \mathcal{Q}$: Let a be a positive integer with $(a, n) = 1$. By Bézout's identity, we can find $b \in \mathbb{N}$ such that $n \mid a^2b+1$. By $\mathcal{P}$, $n \mid a^2 + b$. Then $a^4 - 1 = a^2(a^2 + b) - (a^2b+1)$, so $n \mid a^4 - 1$.

Proof that $\mathcal{Q} \Rightarrow \mathcal{P}$: Let a, b be positive integers with $n \mid a^2b+1$. Clearly $(a, n) = 1$, so $n \mid a^4-1$. Then $a^2(a^2 + b) = (a^4 - 1) + (a^2b + 1)$. Since a and n are relatively prime, $n \mid a^2 + b$.

Now we wish to find all n with property $\mathcal{Q}$. If a is odd, we have $a^4 - 1 = (a^2 - 1)(a^2 + 1)$, $a^2 \equiv 1 \pmod 8$, and $a^2 + 1$ is even, so $16 \mid a^4 - 1$. If $(a, 3) = 1$, we have $a^2 \equiv 1 \pmod 3$, so $3 \mid a^4 - 1$. If $(a, 5) = 1$, we have $5 \mid a^4 - 1$ by Fermat's Little Theorem. This argument shows that $n \mid 240$ is sufficient.

To show $n \mid 240$ is necessary, suppose n has property $\mathcal{Q}$, and let $n = 2^a \cdot k$, where k is odd. If $k > 1$, then $(k - 2, n) = 1$, so by $\mathcal{Q}$ we conclude that $n \mid (k - 2)^4 - 1$. Then $k \mid (k - 2)^4 - 1$, but

$(k-2)^4 \equiv (-2)^4 \equiv 16 \pmod{k}$, so $k \mid 15$. Now, since $(11, n) = 1$, $n \mid 11^4 - 1$, so $2^a \mid 11^4 - 1$, resulting in $a \leq 4$. Thus $n \mid 240$ is also necessary.

The number of natural numbers n such that property $\mathcal{P}$ holds is simply the number of positive integer divisors of 240, which is $(4+1)(1+1)(1+1) = \boxed{20}$.

14. Find constants α and c such that the following limit is finite and nonzero:

$$c = \lim_{n\to\infty} \frac{e\left(1-\frac{1}{n}\right)^n - 1}{n^\alpha}.$$

Give your answer in the form (α, c).

Answer: $(-1, -1/2)$

Solution 1: Take $x = 1/n$ and let $F(x) = e(1-x)^{1/x}$. Then we have

$$F(x) = e(1-x)^{1/x} = \exp\left(1 + \frac{\log(1-x)}{x}\right) = \exp\left(-\frac{x}{2} - \frac{x^2}{3} - \frac{x^3}{4}\cdots\right)$$

since $\log(1-x)$ has Taylor expansion $-x - x^2/2 - x^3/3 - \cdots$. Especially observe that $F(0) = 1$, so when $\alpha = -1$ the limit can be represented as a derivative of F, as follows:

$$\lim_{n\to\infty} \frac{e\left(1-\frac{1}{n}\right)^n - 1}{1/n} = \lim_{x\to 0} \frac{F(x) - F(0)}{x} = F'(0) = -\frac{1}{2}.$$

Solution 2: Observe that if $\alpha \geq 0$ then the limit would be zero, so therefore $\alpha < 0$. This suggests that we should replace $\frac{1}{n} = x$, so the limit becomes

$$c = \lim_{x\to 0} \frac{e(1-x)^{1/x} - 1}{x^{-\alpha}}.$$

Applying L'Hôpital's Rule yields

$$c = \lim_{x\to 0} \frac{e(1-x)^{1/x}\left(\frac{1}{x^2-x} - \frac{\ln(1-x)}{x^2}\right)}{-\alpha x^{-\alpha-1}}.$$

Observe that $\ln(1-x) \approx -x - \frac{x^2}{2}$, and that $\lim_{x\to 0} e(1-x)^{1/x}$, so therefore the numerator is

$$\lim_{x\to 0} e(1-x)^{1/x}\left(\frac{1}{x^2-x} - \frac{\ln(1-x)}{x^2}\right) = \lim_{x\to 0}\left(\frac{1}{x^2-x} - \frac{\ln(1-x)}{x^2}\right)$$
$$= \lim_{x\to 0}\left(\frac{1}{x^2-x} + \frac{x + \frac{x^2}{2}}{x^2}\right) = \lim_{x\to 0}\left(\frac{1}{x^2-x} + \frac{1}{x} + \frac{1}{2}\right) = \lim_{x\to 0}\left(\frac{1}{x-1} + \frac{1}{2}\right) = -\frac{1}{2}.$$

This is nonzero, so therefore our desired α is $\alpha = -1$, with limit $-1/2$, and our final answer is $\boxed{(-1, -1/2)}$.

15. Sean thinks packing is hard, so he decides to do math instead. He has a rectangular sheet that he wants to fold so that it fits in a given rectangular box. He is curious to know what the optimal size of a rectangular sheet is so that it's expected to fit well in any given box. Let a and b be positive reals with $a \leq b$, and let m and n be independently and uniformly distributed random variables in the interval $(0, a)$. For the ordered 4-tuple (a, b, m, n), let $f(a, b, m, n)$ denote the

ratio between the area of a sheet with dimension $a \times b$ and the area of the horizontal cross-section of the box with dimension $m \times n$ after the sheet has been folded in halves along each dimension until it occupies the largest possible area that will still fit in the box (because Sean is picky, the sheet must be placed with sides parallel to the box's sides). Compute the smallest value of $\frac{b}{a}$ that maximizes the expectation f.

Answer: $e^{-1+\sqrt{1+2\ln(2)^2}}$

Solution: First, note that for fixed a, b, m, and n, $f(a,b,m,n) = f(a,b,m/2,n) = f(a,b,m,n/2)$ because we can go from an optimal sheet folding in one case to an optimal sheet folding in another case by either folding or unfolding the sheet in half, which scales the sheet's folded area by the same amount as the box's cross-sectional area. This implies that we can tile the region $m \in (0,a)$, $n \in (0,a)$ with an infinite number of rectangles for which the expectation of f is the same inside each rectangle. In particular, the set R of these rectangles is the set of all rectangles bounded by points of the form $(\frac{a}{2^x}, \frac{a}{2^y})$ and $(\frac{a}{2^{x+1}}, \frac{a}{2^{y+1}})$ in mn-space for nonnegative integers x and y. The expectation of f over the whole region is the same as the expectation of f inside any one of these rectangles. Let us choose to examine the rectangle where we have $m \in [a/2, a)$, $n \in [a/2, a)$. Clearly, it is optimal to fold the sheet exactly once in the dimension where the sheet has length a. For the other dimension, we may assume that $b \in [a, 2a)$ because if $b \geq 2a$, we are forced to fold it anyways until $b < 2a$. Now, we only have to fold in the b dimension once if $b/2 < \max(m,n)$, and otherwise we must fold twice. Therefore, the expected value of f is equal to

$$\frac{4}{a^2}\int_{a/2}^{a}\int_{a/2}^{a} \frac{(a/2)(b/2)(1/2)^{H(b/2-\max(m,n))}}{mn}\,dm\,dn,$$

where $H(x)$ is the Heaviside Step Function (defined as $H(x) = 1$ if $x \geq 0$ and 0 otherwise).

We can compute this integral by noticing that it is equal to

$$\frac{4}{a^2}\left(\int_{a/2}^{a}\int_{a/2}^{a} \frac{ab/4}{mn}\,dm\,dn - \int_{a/2}^{b/2}\int_{a/2}^{b/2} \frac{ab/8}{mn}\,dm\,dn\right).$$

Hence, this problem relies on evaluating

$$\int_{c}^{d}\int_{c}^{d} \frac{1}{xy}\,dx\,dy = \int_{c}^{d} \frac{\ln(d)-\ln(c)}{y}\,dy = (\ln(d)-\ln(c))^2 = \ln(d/c)^2.$$

Plugging in, we get that the original integral equals

$$\frac{4}{a^2}((ab/4)\ln(2)^2 - (ab/8)\ln(b/a)^2) = \frac{b}{2a}(2\ln(2)^2 - \ln(b/a)^2).$$

Let $k = b/a$, so that the expectation we are maximizing is $g(k) = \frac{k}{2}(2\ln(2)^2 - \ln(k)^2)$ over the domain $k \in [1,2)$. The derivative of g is

$$\frac{1}{2}(2\ln(2)^2 - \ln(k)^2) + \left(\frac{-2\ln(k)}{k}\right)\left(\frac{k}{2}\right) = -\frac{1}{2}(\ln(k)^2 + 2\ln(k) - 2\ln(2)^2),$$

which we set to zero, getting

$$\ln(k) = \frac{-2 \pm \sqrt{4+8\ln(2)^2}}{2} = -1 \pm \sqrt{1+2\ln(2)^2}.$$

We now need to check two final things. First, we must see if either of these solutions to $g'(k) = 0$ are in the interval $(1, 2)$. Only the positive solution to the above quadratic could possibly result in k being in greater than 1 to begin with. The easiest way to see that this gives us a value in the interval $(1, 2)$ is by noticing that $g(1) = g(2) = \ln(2)^2$ (since the $k = 1$ and $k = 2$ cases both result in the folded sheet having the same area, namely $a/2 \times a/2$, for all choices of m and n), so we are guaranteed a point with zero derivative in the interval $(1, 2)$ by Rolle's Theorem.

Additionally, we must check that this is a local maximum and not a minimum. We claim here that $g(k)$ is concave down on the interval $(1, 2)$, so what we have found is a local maximum. First, $\ln(k)^2$ is concave up on $(1, 2)$ because

$$\frac{d^2}{dk^2}\ln(k)^2 = \frac{2 - 2\ln(k)}{k^2},$$

which is positive when $\ln(k) < 1 \iff k < e$. Hence, $2\ln(2)^2 - \ln(k)^2$ is concave down in the same interval. It is also clearly decreasing. Finally, we have for generic functions f and g that if $f(x) = xg(x)$, then $f'(x) = g(x) + xg'(x)$ and $f''(x) = 2g'(x) + xg''(x)$, so on an interval where g is decreasing and concave down and x is positive, then f is guaranteed to also be concave down. This scenario holds for our function $g(k)$, so it is concave down.

Hence, report $\boxed{e^{-1+\sqrt{1+2\ln(2)^2}}}$.