SMT Power Round Solutions : Poles and Polars

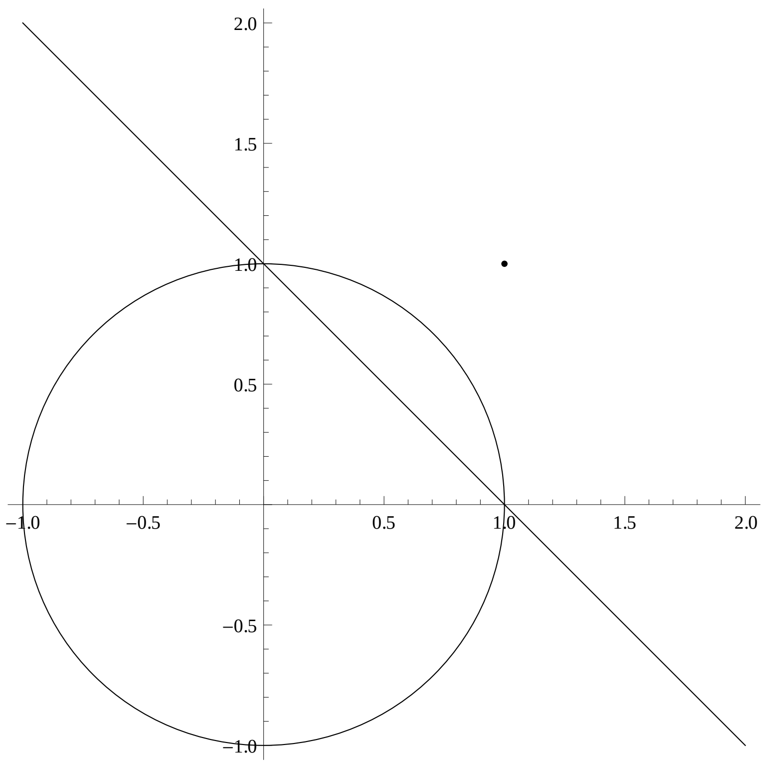
February 18, 2011

1 Definition and Basic Properties

1

Note that the unit circles are not necessary in the solutions. They just make the graphs look nicer.

(1)



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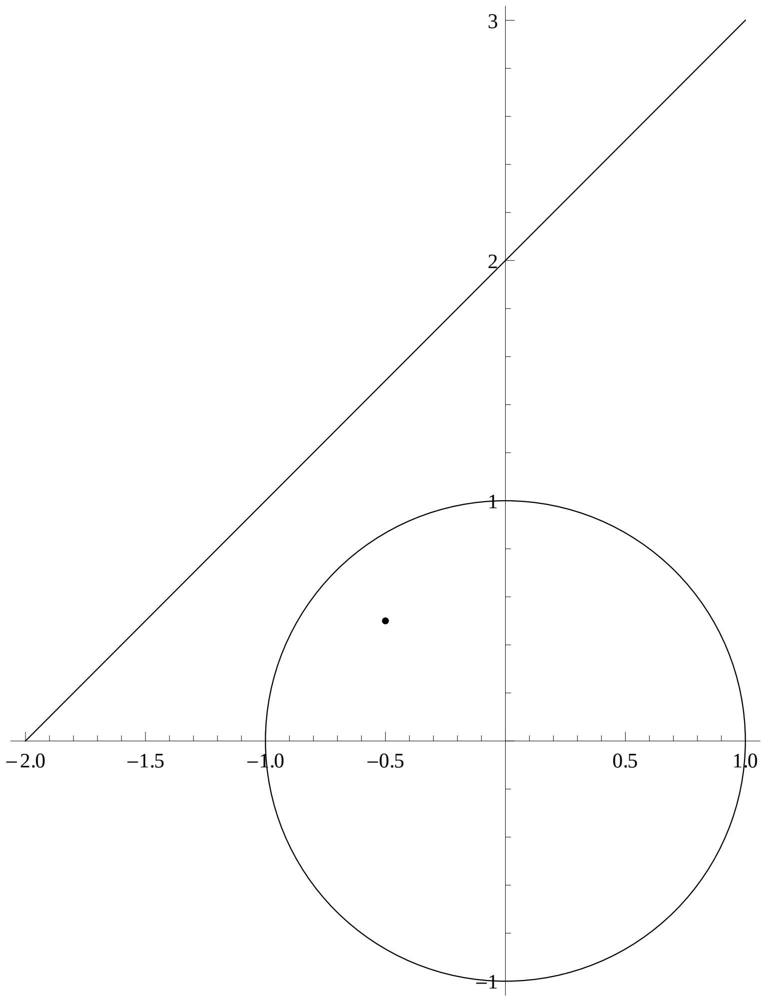

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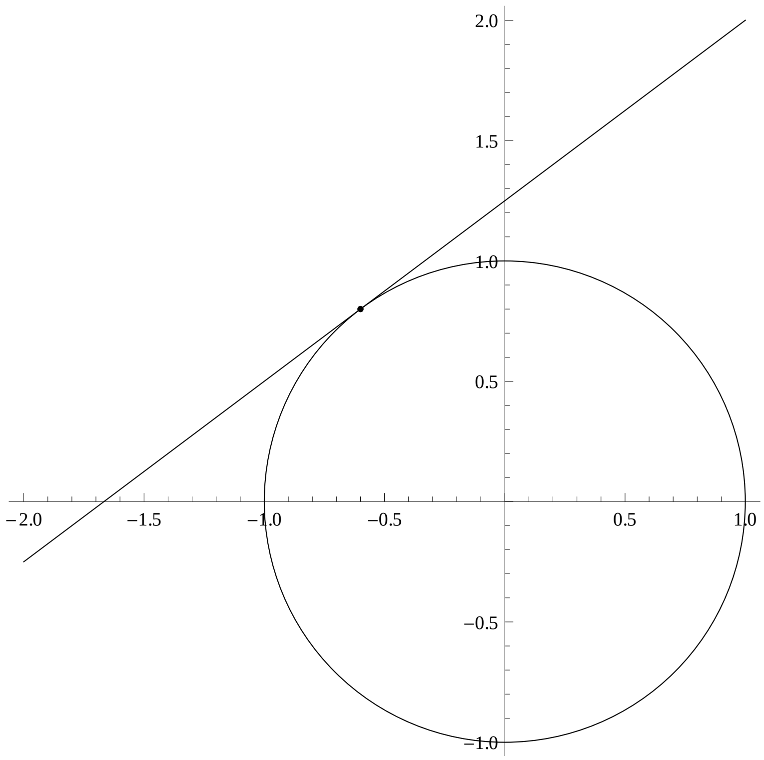

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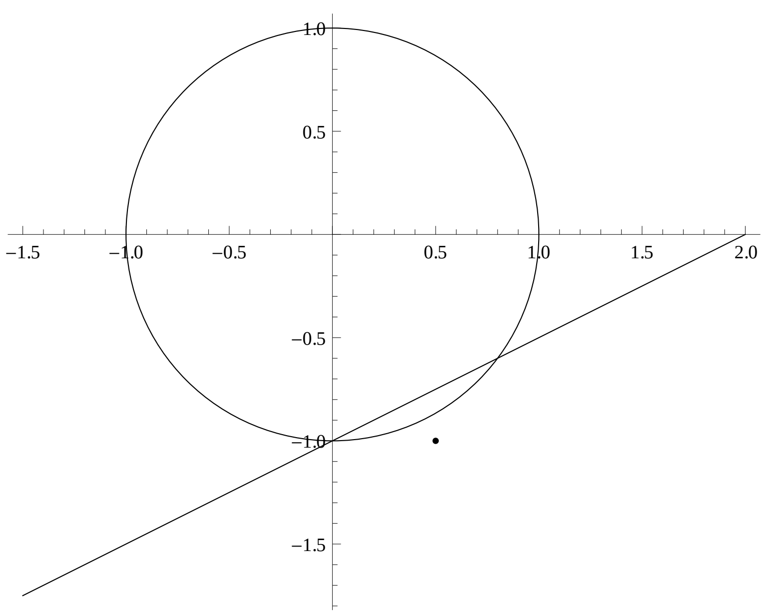
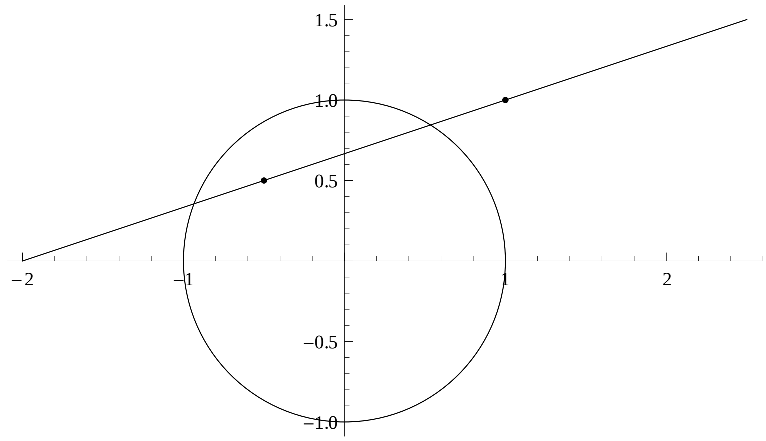


(5)

This is the same as in (1).

(6)

It is interesting to note that this line goes through the A and the pole of b.



2

The perpendicular through the origin to the polar of P is $-p_y x + p_x y = 0$. This line intersects P at $P^* = \frac{P}{|P|^2}$. In particular, $|P^*| = \frac{1}{|P|}$. So P^* is on the other side of the unit circle from P. Since P^* is the closest point on the polar to the origin, the polar intersects the unit circle once when P^* is on the unit circle, twice when P^* is inside the unit circle, and zero times when P^* is outside the circle. In other words, the polar intersects the unit circle once when P is on the unit circle, twice when P is outside the unit circle, and zero times when P is inside the unit circle.

3

Let $P' = \frac{P}{|P|^2}$. This is the inversion of P because it is clearly on the ray $\overline{OP}$, and also because it satisfies $\overline{OP} \cdot \overline{OP'} = \frac{\overline{OP} \cdot \overline{OP'}}{|P|^2} = 1$.

P' is on the polar of P because $p_x(\frac{p_x}{p_x^2+p_y^2}) + p_y(\frac{p_y}{p_x^2+p_y^2}) = 1$. Finally, the polar is perpendicular to $\overline{OP}$ because $\overline{OP}$ is pointing in the same direction as the normal (p_x, p_y) to the polar.

4

Let $O = (a, b)$ and apply the definition from problem 3 except replace $\overline{OP} \cdot \overline{OP'} = 1$ with $\overline{OP} \cdot \overline{OP'} = r^2$. In particular, let's find the point $P' = (p'_x, p'_y)$ on the ray $\overline{OP}$ such that $\overline{OP} \cdot \overline{OP'} = r^2$. To do this, parametrize the ray $\overline{OP}$ as $R(t)$ where

$$R(t) = (a + (p_x - a)t, b + (p_y - b)t),$$

for $t \geq 0$. Then notice that

$$\overline{OP} \cdot \overline{OR(t)} = p_x(p_x - a)t + p_y(p_y - b)t$$

Setting this to r^2 and solving for t gives $t = \frac{r^2}{p_x^2+p_y^2-ap_x-bp_y}$. Plugging this into our parametrization,

$$P' = R\left(\frac{r^2}{p_x^2 + p_y^2 - ap_x - bp_y}\right) = \left(a + \frac{(p_x - a)r^2}{|OP|^2}, b + \frac{(p_y - b)r^2}{|OP|^2}\right).$$

Finally, note that the perpendicular to $\overline{OP}$ through $P' = (p'_x, p'_y)$ is given by

$$(p_x - a)(x - p'_x) + (p_y - b)(y - p'_y) = 0.$$

Plugging in our coordinates for P' gives the answer, which is

$$(p_x - a)x + (p_y - b)y = (p_x - a)a + (p_y - b)b + r^2.$$

2 The Duality Principle

5

Suppose $A = (x_A, y_A)$ is on the polar of $B = (x_B, y_B)$. The polar of A is $x_A x + y_A y = 1$. Since B is on this line, $x_A x_B + y_A y_B = 1$. This immediately implies that A is on the polar $x_B x + y_B y = 1$ of B.

The other direction is completely symmetric.

Alternatively, use the definition from problem 3. Let A' and B' be the inversions of A and B respectively, and suppose B is on the polar of A. Consider the point B'' which is the foot of the perpendicular from A to OB. Since $\angle AA'B = \angle AB''B = 90°$, quadrilateral $AA'B''B$ is cyclic, and by Power of a Point on its circumscribed circle and point O, we have $(OB'')(OB) = (OA')(OA) = r^2$ where r is the radius of circle O. But then $OB'' = r^2/OB = OB'$, so in fact $B' = B''$. Therefore A lies on the perpendicular to OB through B', otherwise known as the polar of B.

6

Let $A = (x_A, y_A)$, $B = (x_B, y_B)$, and $C = (x_C, y_C)$. Then a is $x_A x + y_A y = 1$, b is $x_B x + y_B y = 1$, and c is $x_C x + y_C y = 1$.

a

This is just a restatement of 5, in our new notation.

b

The intersection of a and b is a point that is on both polars a and b. So by part (a), both A and B lie on the polar of the intersection of a and b. In other words, the polar of the intersection of a and b is the line AB.

The converse follows immediately from the fact that reciprocation is an involution. In particular, apply reciprocation to the statement "the polar of the intersection of a and b is the line AB" to get the statement "the intersection of a and b is the polar of AB."

c

Suppose a, b, c go through the same point. Then, by part (a), the polar of this point goes through A, B, C. In particular, A, B, C are collinear.

Suppose A, B, C are collinear. Then the pole of the line through A, B, C is on a, b, c. In particular, a, b, c all go through the same point.

7

Given a, b, c concurrent and d, e, f concurrent, the three lines $x = (b \cap f)(c \cap e)$, $y = (a \cap f)(c \cap d)$, and $z = (a \cap e)(b \cap d)$ are also concurrent.

8

Take the dual to be the dual around the relevant circle, as described in our generalization in (5). Then incidence of a point with a circle corresponds with tangency of its polar with the circle.

Let $abcdef$ be sides of a cyclic dual hexagon (not necessarily in that order). Extend them to lines $abcedf$. Then the three lines $(a \cap b)(d \cap e)$, $(b \cap c)(e \cap f)$, and $(c \cap d)(f \cap a)$ are concurrent.

3 Reciprocation and Cyclic Quadrilaterals

9

1

$$X = \left(\frac{n}{m+n} x_a + \frac{m}{m+n} x_b, \frac{n}{m+n} y_a + \frac{m}{m+n} y_b \right)$$

$$Q = \left(\frac{-n}{m-n} x_a + \frac{m}{m-n} x_b, \frac{-n}{m-n} y_a + \frac{m}{m-n} y_b \right)$$

2

Just plug in and check

$$\left(\frac{n}{m+n} x_a + \frac{m}{m+n} x_b \right) \left(\frac{-n}{m-n} x_a + \frac{m}{m-n} x_b \right) + \left(\frac{n}{m+n} y_a + \frac{m}{m+n} y_b \right) \left(\frac{-n}{m-n} y_a + \frac{m}{m-n} y_b \right)$$
$$= \frac{-n^2}{m^2-n^2}(x_a^2 + y_a^2) + \frac{m^2}{m^2-n^2}(x_b^2 + y_b^2) = \frac{-n^2}{m^2-n^2} + \frac{m^2}{m^2-n^2} = 1$$

3

Apply Menelaus' theorem to the triangle XYQ and the collinear points AP_1C, giving

$$\frac{XP_1}{P_1Y} \frac{YC}{CQ} \frac{QA}{AX} = -1 \implies \frac{XP_1}{P_1Y} = -\frac{CQ}{YC} \frac{AX}{QA}$$

Apply Menelaus' theorem to the triangle XYQ and the collinear points BP_1D, giving

$$\frac{XP_2}{P_2Y} \frac{YB}{BQ} \frac{QD}{DX} = -1 \implies \frac{XP_2}{P_2Y} = -\frac{BQ}{YB} \frac{DX}{QD}$$

The two ratios we just computed are the same because of the equations $\frac{AX}{XD} = \frac{AQ}{QD}$ and $\frac{BY}{YC} = \frac{BQ}{QC}$ defining X and Y respectively.

Therefore $P_1 = P_2$. And since $P_1 = P_2$ is both on AC and BD, and P is defined as the intersection of AC and BD, $P = P_1 = P_2$.

4

X and Y are both on the polar of Q by part (2). So the line XY is the polar of Q. By part (3), P is on XY. So P is on the polar of Q.

10

What we proved in part (9) is that if we have four points A, B, C, D on a circle, then $AD \cap BC$ is on the polar of $AC \cap BD$. Just permute the points in this statement around to see that, in fact, each pair of points in P, Q, R is on the polar of the other point.

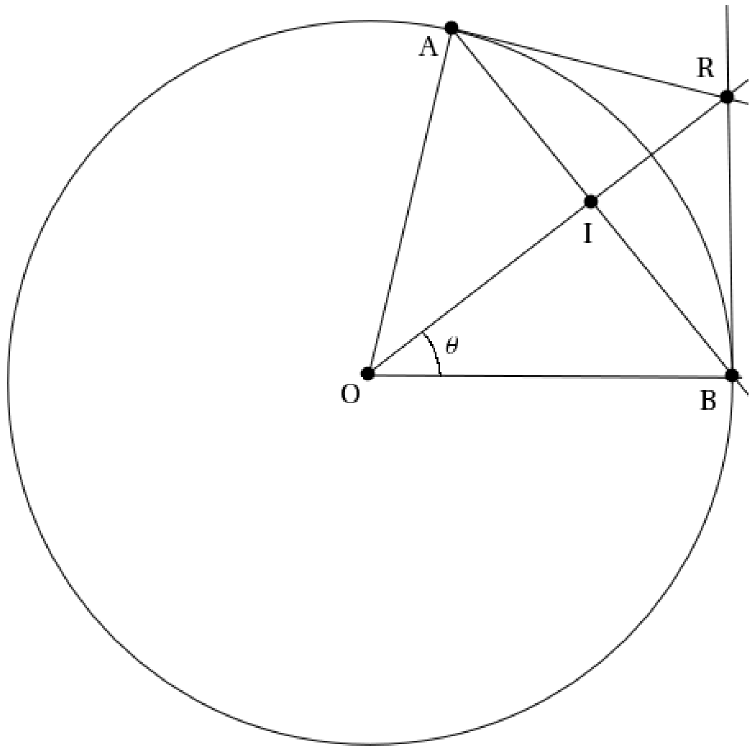
The orthocenter of PQR is the center of the circle. In particular, if we are using the unit circle around the origin, then the orthocenter of PQR is the origin. This is true because each side of the triangle is the polar of the opposite vertex. So the perpendicular through each side through the opposite vertex goes through the center of the circle. Ie, all three altitudes intersect at the center of the circle.

11

We already know that (1) and (2) are on the polar of $AB \cap CD$, so let's check the remaining points.

Point (3). Observe the following diagram. We want to show that R, the intersection of the tangent at A and the tangent at B is on the polar of $AB \cap CD$. Equivalently, we need to show that $AB \cap CD$ is on the polar of R. To show this, let's show that AB is the polar of R.

First, note that AB is perpendicular to OR by symmetry. So AB is parallel to the polar of R. To see that AB is in fact equal to the polar of R, we need only show that $OI \cdot OR = 1$. One easy way to see this is to note that $|OI| = \cos\theta$ and $|OR| = \frac{1}{\cos\theta}$.



Point (4). Same as point (3).

Point (5). Call point (5) X. We want to show that X is on the polar of $AB \cap CD$, which is equivalent to showing that $AB \cap CD$ is on the polar of X. Since X is on the circle, the polar of X is the tangent to the circle at X. By the definition of X, $AB \cap CD$ is on this tangent. So we are done.

Point (6). Same as point (5).

12

The line through P and the other intersection of the circumcircles is the radical axis of the two circumcircles. So it suffices to prove that another of the "six points" also lies on the radical axis. But by Power of a Point on the circle in which $ABCD$ is inscribed, $AD \cap BC$ clearly has the same power with respect to both circles. Therefore the second intersection of the circumcircles indeed lies on the line.

4 Conic Sections

13

(1) It's an ellipse. Every tangent to circle A is at distance at least $r - OA$ from O, and thus every pole of such a tangent is at distance at most $1/(r - OA)$ from O—that is, the locus of these poles is bounded. Therefore it must be an ellipse.

(2) The polars of the points of circle A are tangents to the ellipse by duality.

(3) The conic section is a parabola, since there is precisely one tangent line (the one at O) which has no finite pole, so that the conic should go to infinity along AO and be continuous elsewhere.

(4) The conic section is a hyperbola. There are now two tangents to circle A passing through O, neither of which have a pole. Their points of tangency to A have polars which are tangent to the conic but never touch it. A conic section with two asymptotes is a hyperbola.

14

As previously explained, each tangent to the parabola is the polar of a point T on circle A. The foot of the perpendicular from O to the tangent is the inversion of T with respect to circle O. So the locus is the inversion of circle A about circle O, which is a line. Specifically, it's the line going through the intersection of the circles.

15

Let P be a point on the conic and let TM be its polar where T is the point of tangency to circle A. The inversion P' of P lies on TM. Since M lies on the polar of P, P lies on the polar of M, so that the foot of the perpendicular from P to line OA is the inversion M' of M. Let A' be the inversion of A and K be the foot of the perpendicular from P to a. We need to prove that $OP = \epsilon PK$. We will use directed distances to avoid casework. We calculate

$$\frac{PK}{OP} = \frac{OA' - OM'}{OP} = OP'\left(\frac{1}{OA} - \frac{1}{OM}\right)$$

$$= \frac{OP'}{OM}\left(\frac{OM}{OA} - 1\right) = \frac{AT}{AM}\frac{AM}{OA} = \frac{r}{OA}.$$

We set $\epsilon = \frac{r}{OA}$. (Note: this section was mostly taken from Geometry Revisited.)

5 Counting

16

First we prove by induction the nice fact that n lines split the plane into a maximum of $\frac{n(n+1)}{2} + 1$ regions.

For the base case, notice that 1 line divides the plane into $2 = 1 + 1$ regions.

For the inductive step, assume that $n-1$ lines divide the plane into a maximum of $\frac{n(n-1)}{2}+1$ regions. Add an n-th line that intersects all $n-1$ lines in points where they are not intersecting each other. This new line splits each of the n regions it goes through into 2 regions. Ie, this new line adds n regions. So we have $\frac{n(n-1)}{2}+1+n = \frac{n(n+1)}{2}+1$ regions.

We still need to show that we can't get more than $\frac{n(n+1)}{2}$ regions with n lines. If we do get more regions, then either (1) we started with more than $\frac{n(n-1)}{2} + 1$ regions before we added the n-th line or (2) we added more than n regions when we added the n-th line. But (1) contradicts the inductive hypothesis that $n-1$ lines give us a maximum of $\frac{n(n-1)}{2} + 1$ regions. And if (2) is the case, then our n-th line split more than n regions, forcing it to have intersected more than $n-1$ old lines. But there are only $n-1$ old lines to intersect. So we are done.

Next, we claim that the maximum number of distinct linear partitions is $\frac{n(n-1)}{2} + 1$ and we use our lemma to prove this.

Take any set of n points $P_1, ..., P_n$ and translate them so that P_n is at the origin. Let $L_1, ..., L_{n-1}$ be the dual lines of $P_1, ..., P_{n-1}$ and let $R_1, ..., R_s$ be the regions bounded by these lines. Now we will show that there is a one-to-one correspondence between the regions $R_1, ..., R_s$ and the linear partitions of $P_1, ..., P_n$.

To get a linear partition from a region R_i, take a point $X \in R_i$. The dual of X is some line ℓ that gives us a linear partition of $P_1, ..., P_n$. This partition is independent of our choice of $X \in R_i$ because as we move X around in R_i we don't touch any of the dual lines $L_1, ..., L_{n-1}$ and therefore ℓ never goes through any of the points $P_1, ..., P_n$. Furthermore, we get every partition because every partition induced by a line ℓ is achieved when we take X to be the dual point to ℓ. Finally, each partition we get is different because when we choose a region R on one side of a line L_j we get P_j being in the same side of the partition as the origin and when we choose a region R on the other side of the line L_j we get P_j being on the other side of the partition as the origin.

So there are the same number of linear partitions of $P_1, ..., P_n$ as there are regions created by the $n-1$ dual lines to $P_1, ..., P_{n-1}$. By our lemma, we can arrange for this to be $\frac{n(n-1)}{2} + 1$ but we can't have it be any bigger. So $\frac{n(n-1)}{2}$ is the maximum.